

On emergence in gauge theories at the 't Hooft limit

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Abstract

Quantum field theories are notoriously difficult to understand, physically as well as philosophically. The aim of this paper is to contribute to a better conceptual understanding of gauge quantum field theories, such as quantum chromodynamics, by discussing a famous physical limit, the 't Hooft limit, in which the theory concerned often simplifies.

The idea of the limit is that the number N of colours (or charges) goes to infinity. The simplifications that can happen in this limit, and that we will consider, are: (i) the theory's Feynman diagrams can be drawn on a plane without lines intersecting (called 'planarity'); and (ii) the theory, or a sector of it, becomes integrable, and indeed corresponds to a well-studied system, viz. a spin chain. Planarity is important because it shows how a quantum field theory can exhibit extended, in particular string-like, structures; in some cases, this gives a connection with string theory, and thus with gravity.

Previous philosophical literature about how one theory (or a sector, or regime, of a theory) might be emergent from, and-or reduced to, another one has tended to emphasize cases, such as occur in statistical mechanics, where the system before the limit has finitely many degrees of freedom. But here, our quantum field theories, including those on the way to the 't Hooft limit, will have infinitely many degrees of freedom.

Nevertheless, we will show how a recent schema by Butterfield and taxonomy by Norton apply to the quantum field theories we consider; and we will classify three physical properties of our theories in these terms. These properties are planarity and integrability, as in (i) and (ii) above; and the behaviour of the beta-function reflecting, for example, asymptotic freedom.

Our discussion of these properties, especially the beta-function, will also relate to recent philosophical debate about the propriety of assessing quantum field theories, whose rigorous existence is not yet proven.

Contents

1	Prospectus	2
2	At the limit vs. before the limit, in gauge theories	6
2.1	A schema and a trichotomy	6
2.2	The schema and trichotomy in QCD and SYM	10
2.2.1	Do gauge theories exist? Should we dive in?	11
2.2.2	How the properties will be classified	17
3	The ‘t Hooft limit in QCD	19
3.1	The need for an approximation scheme	19
3.2	The emergence of planarity	22
3.2.1	Conceptual remarks	23
4	Introducing super Yang-Mills theory	25
4.1	Introducing the theory	25
4.2	The vanishing β -function and conformal invariance	26
4.3	Dilatation and primary operators	27
4.4	Gauge invariant operators and a Fock space	29
5	Integrability at the limit, and the relation to spin chains	30
5.1	Computing anomalous dimensions	31
5.1.1	Planar diagrams dominate	31
5.1.2	The operator Γ	33
5.2	Spin chains	34
6	Aspects of non-integrability before the limit	37
7	Conclusion	38

1 Prospectus

Physical theories often simplify, and so can be better understood, in some limit. A familiar example is the thermodynamic limit in statistical mechanics, which takes the number N of constituents to infinity (while actually $N \approx 10^{23}$). Here, we hope to contribute to a better conceptual understanding of the heuristic formulation of gauge quantum field theories, by discussing their ‘t Hooft limit: in which the theory concerned often simplifies.

This simplification is very surprising because the limit is defined, for quantum chromodynamics (QCD) and other gauge theories, by taking the number N of colours,

which in QCD is actually three, to infinity: thus passing from $N = 3$ through apparently very complex theories, with say googol (10^{100}) colours. Just as in thermodynamics the limit $N \rightarrow \infty$ is “controlled” by requiring that a quantity like the density remain fixed (so that the volume also goes to infinity): so also here, the limit is controlled by requiring conditions on the product $\lambda := g^2 N$ (of the square of the coupling constant g with N). λ is called ‘the ’t Hooft coupling’, and the condition is ‘being fixed’.

We will connect the physics of these limits to philosophical discussions of one theory (or a sector, or regime, of a theory) being emergent from, and-or reduced to, another one. Besides, our physical and philosophical claims are independent of much of the details of QCD. They apply to other non-abelian gauge fields, i.e. with a gauge group different from QCD’s $SU(N)$, e.g. $SO(N)$ or $U(N)$.

The ’t Hooft limit is a rich subject, with many aspects which we cannot pursue here. One main one, which we will study in a companion paper, is that this limit sheds light on the connection between quantum field theories and gravity—a connection much studied under the label ‘AdS/CFT’. Here we will instead be concerned with three aspects, occurring in two theories, viz. QCD and a supersymmetric cousin of it. These aspects are:

(1): In both theories, we will be concerned with the high-energy behaviour of interaction strengths as described by the beta-function, i.e. asymptotic freedom or conformal invariance (Sections 3.1 and 4.2). As we will discuss in Section 2.2.1, this aspect is central to the question whether the theory rigorously exists.

(2): We will discuss emergent planarity and related string-like structures, first for QCD (Section 3.2) and then for the supersymmetric cousin (Section 5.1). This aspect relates closely to the string-gravity connection, AdS/CFT; and to the appearance of a classical regime.

(3): We will discuss emergent integrability, in the supersymmetric cousin of QCD (Sections 4.3 to 5.2). This aspect relates closely to whether the theory is tractable for calculation and-or mathematical analysis.

Since in the ’t Hooft limit, the parameter N is the number of colours, the situation is different from those considered in philosophical discussions of statistical mechanics. There, the parameter is often the number of degrees of freedom, or a similar notion like the number of constituent particles; while here, in field theories, the number of degrees of freedom is infinite.

But despite this difference, our three aspects (1)-(3) will fit in with those discussions. In particular, we will classify the aspects using a recent schema of Butterfield’s and a taxonomy of Norton’s, about the degree of “meshing” between what holds good at the limit $N = \infty$, and what holds good before the limit, at finite N . This meshing makes the limit valuable as a framework for studying the finite N theories: which are those of physical interest, since no one believes that the number of colours is in fact infinity. Thus one even envisages making an expansion in $1/N$ about the limiting theory, and then—again: very surprisingly—getting a physical result by substituting

in the actual (finite!) value of N , even a very small one like 3.

We now turn to a Section-by-Section prospectus.

Section 2 sets the scene and includes a summary (Section 2.2.2) of how our three aspects (1)-(3) will be classified in our two theories. So the reader in a hurry can even stop after Section 2. In more detail: in Section 2.1, we will: (i) present Butterfield's schema for inter-theoretic relations, which allows emergence, i.e. novelty of theoretical structures, to be combined with reduction, by taking a limit; (ii) present Norton's taxonomy—in fact, a trichotomy—of degrees of meshing at the limit; and then (iii) compare the schema and trichotomy. In Section 2.2, we will discuss: (i) whether our theories, and others like them, rigorously exist; (ii) how this issue is affected by two remarkable properties of the theories, viz. asymptotic freedom and conformal invariance; and (iii) how this issue affects the philosophical discussion. Then, using these discussions' conclusions, Section 2.2.2 sums up how our three aspects (1)-(3) will be classified.

Then in Section 3, we sketch some relevant aspects of gauge theories, especially QCD. The main point will be the surprising simplification. QCD is described by a gauge group $SU(3)$, where 3 is the number of colours: the gauge fields are described by 3×3 matrices. So one naturally expects that theories using a value of N higher than 3, i.e. using larger matrices, will be increasingly complex. But in 1974, 't Hooft discovered that the theory simplifies at $N = \infty$. In the perturbation (Feynman-diagram) expansion, only those diagrams that can be drawn on a two-dimensional plane without intersecting lines (called 'planar diagrams') remain. This is associated with the appearance of string-like structures; and with the perturbation expansion being in a certain sense a complete representation of the theory.¹

In Section 4, we describe the novel features and mathematical structures that can occur in the $N = \infty$ limit. But we 'change horses', i.e. we consider a different theory. For in this limit, QCD remains a complicated theory. So we emphasize, as the physics literature does, a simpler theory, again in this limit: called 'maximally supersymmetric Yang-Mills theory'. Despite the long name, this is simpler than QCD! That is: the details of its dynamics are easier to study. We will dub it 'SYM'.² SYM is also guessed and hoped to be one of the essential models for understanding more complicated gauge theories, including QCD itself. So much so that nowadays SYM is sometimes called 'the harmonic oscillator of quantum field theory'.

¹As we shall see, the main rationale for focussing on QCD is that—apart from the historical point that 't Hooft introduced his limit for QCD—it is asymptotically free; and this is related to the theory's rigorously existing (cf. respectively Section 3.1 and Section 2.2).

²As we shall see in Sections 4.1 and 4.2, this simplicity arises from a high degree of symmetry, especially conformal invariance as given by a vanishing beta-function. Besides, field theories with a supersymmetry which is less than maximal (i.e. less than in SYM) have been shown to be in some respects easier to study than non-supersymmetric theories, and in particular to exhibit analytically important features of quantum gauge theories, such as confinement and a mass gap: e.g. Seiberg and Witten (1994).

We emphasize that our discussion of SYM does not require that nature actually be supersymmetric; just as our discussing the 't Hooft limit does not require that nature actually have infinitely many colours. More positively, we believe that foundational work can, and even should sometimes, consider theories because they shed light on conceptual questions and-or are analytically tractable: and we submit that both these reasons justify considering the 't Hooft limit of QCD and SYM. Indeed, these reasons also justify studying supersymmetry more generally. A good example of this is the beta-function itself: although in QCD we can only calculate this perturbatively, there are some supersymmetric theories whose beta-function has been computed exactly (Novikov et al. (1983), Shifman (2012, pp. 531-533)). (Of course, considering such theories echoes the widespread acceptance of the Wilsonian perspective on renormalization, with its use of a space of theories, connected by the renormalization group flow.)

In Section 5, we arrive at integrability in the 't Hooft limit. That is: we discuss how, thanks to planarity, certain physical aspects of SYM are mapped into integrable spin chains. Here we connect with an old dream of quantum field theory: to calculate analytically the mass-spectrum of a theory, i.e. the masses of its particles such as a proton, as a function of the parameters of the theory, such as coupling constants and the energy-scale. SYM is conformally invariant: that is, roughly speaking, invariant under a transformation that changes scales but preserves angles. This makes the theory have no massive particles. But there is an analogue of the mass that one can aspire to compute: namely, the scaling dimension of local operators. (It is also an even closer analogue of the critical exponents occurring in statistical mechanics' description of phase transitions.) The idea is that the correlation function of such an operator, i.e. the correlation (the average of the product) of its values at two different spacetime points (as given by an expectation value of a product), falls off with some power Δ of the spatiotemporal distance between the two points. This Δ is the scaling dimension: it includes a quantum addition (called 'anomalous dimension') to the term Δ_0 obtained by classical dimensional analysis. And it has recently been shown that Δ can be calculated by analysing the associated spin chain systems: a special case of the old dream.³

Finally in Section 6, we briefly discuss the integrability of SYM *before* the $N = \infty$ limit. Considering the situation at finite N is important, because someone might object to the results of Section 5: 'All very well, but any correct quantum field theory of the actual world surely has finite N .' This echoes a familiar objection often urged by philosophers of statistical mechanics about the appeal to the thermodynamic

³This hope is called the 'old dream' at the start of the excellent recent review of integrability in this context, by Beisert et al. (2012), to which (together with Minahan (2012)) our Sections 4 and 5 are indebted: cf. also Polyakov (1987, Chapters 8-10). Although we have postponed to our companion paper discussion of string theory as a theory of gravity, we should stress that these recent advances arose from examining the connection between quantum field theories and string theory, more specifically AdS/CFT.

limit, in which the number of constituent particles is taken to be infinity, in order to explain phase transitions. In effect, they object: ‘my kettle has finitely many atoms in it!’. There are of course parallel misgivings about other explanations using infinitary models of finite systems.

We believe the main answer to this objection is that even before the limit, there is a version of the phenomenon at issue: in the familiar discussions, phase transitions—and here, asymptotic freedom, conformal invariance, planarity and string-like structures. That version may be “weaker” or “approximate”, in various senses, by comparison with the phenomenon at the limit. But it is nevertheless real. We also take this answer to be uncontroversial: we think it is part of how physicists construe such infinite idealizations—albeit usually implicitly, rather than explicitly. We discuss this further in Sections 2.1 and 2.2.

But we should admit here that the situation for integrability is a bit different. As we shall see in Section 6, not much is at present known about integrability at finite N , i.e. on the way to the ‘t Hooft limit. So for physics, this is a straightforward case of work for the future. For philosophy, it illustrates the point that philosophical morals drawn from physics need to beware of the limitations of the present state of knowledge. Finally, Section 7 mentions two further topics to be pursued elsewhere.

2 At the limit vs. before the limit, in gauge theories

2.1 A schema and a trichotomy

Before embarking on the details, we should connect what follows with recent philosophical discussion of inter-theoretic relations, especially the contrast between *at* the limit and *before*, or *on the way to*, the limit. Perceptive recent work includes Callender and Menon (2012), who discuss phase transitions; and Norton (2012) who discusses examples from both thermal physics and geometry. We will connect just with the trichotomy proposed in Norton’s paper: which, we claim, will develop a schema from our own previous discussions (Butterfield (2011, especially Section 3.1-3.2, pp. 1073-1076), Bouatta and Butterfield (2011)).

2.1.A The schema We proposed a schema and a mnemonic notation, for inter-theoretic relations. We wrote T_b for the ‘better, bottom or basic’ theory, and T_t for the ‘tainted or top’ theory. The schema was that (i) in some cases T_t is deduced from T_b (taken together with suitable auxiliary definitions), in some limit of a parameter; and (ii) although deduced, T_t exhibits novel yet robust properties compared with those in T_b . Thus phase transitions illustrate the schema: with T_b taken as the statistical mechanics of N constituents; T_t as thermodynamics, taken as describing phase transitions in terms of singularities of thermodynamic quantities; and with the

limit being the thermodynamic limit, $N \rightarrow \infty$. Butterfield (2011, Section 1, pp. 1066-1071) argued that this schema’s combining (i) and (ii) amounted to reconciling reduction (taken as deduction *à la* (i)) with emergence (taken as novelty *à la* (ii)).

This paper will illustrate the same schema, in the following way. T_t will be either QCD or SYM (or even another gauge theory) in the ’t Hooft limit. The two main novel yet robust properties (of course related to each other) are: string-like structures as shown by planarity of Feynman diagrams (cf. Section 3.2, and Section 5.1.1), and integrability using spin chains (Section 5.2). T_b is of course QCD or SYM at finite N (say, $N = 3!$). Thus again, we will see reduction and emergence reconciled.⁴

2.1.B The trichotomy Norton’s recent analysis (2012) develops this schema. He is concerned to contrast three cases concerning the limit of a theory as a parameter N goes to infinity. We will first report his trichotomy of cases (2012, Sections 3.1-3.3), and compare it with our schema. Then in Section 2.2, we summarize how the three aspects of QCD and SYM that concern us are classified by his trichotomy, and how this sheds light on conceptual aspects of these gauge theories. In later Sections, this summary will of course get fleshed out.

The framework of discussion is that a physical theory describes systems, and their properties, in terms of physical quantities defined on the systems, and states which assign these quantities values (in quantum theories, expectation values). As the parameter N varies from one system to another, there can be more or less “meshing” of the values of corresponding quantities on the systems (for an appropriate choice of states). And one can ask whether there is a mathematically well-defined system at the limit $N = \infty$. In Norton’s examples, N is the number of components, in some natural sense of ‘component’; e.g. the number of squares in a geometric lattice, or the number of atoms (or constituent particles) in a (model of) a sample of gas. So he illustrates $N \rightarrow \infty$ by examples such as ever-finer lattices in geometry, or thermal physics’ thermodynamic limit. Indeed, these examples are topical: philosophical discussion of limiting relations between theories often takes the parameter N to be the number of components, in some natural sense.

Given this framework, we can now state Norton’s three cases. In his first case, systems and properties “mesh” in the limit, as follows:

(1): *Limit property and limit system agree*: There is a well-defined infinite- N system, $\sigma(\infty)$ say. It is usually obtained as a limit of the state spaces of finite, i.e. finite- N , systems $\sigma(N)$. And there is a property of the finite systems, usually given as the value of a quantity, that tends to a limit. Writing $f(N)$ for the quantity on the finite system (and so as short for $f(\sigma(N))$), one might write the value as $v(f(N))$. (Here an appropriate sequence of states s_N is of course understood: $v(f(N))$ is short for $v(f(N), s_N)$). And this limit is the value of a well-defined quantity, $f(\sigma(\infty))$ say, on $\sigma(\infty)$ (where again, a limiting state s_∞ is understood); and this quantity is itself

⁴Our companion paper on AdS/CFT discusses the corresponding theme, that gravity can be both reduced to and emergent from a gauge quantum field theory.

a limit, in some natural sense, of the corresponding quantities on the finite systems. So in our notation: $\lim_{N \rightarrow \infty} v(f(N)) = v(f(\sigma(\infty)))$.

In Norton's second case, the meshing breaks down in that there is no infinite system; while in the third case, there is an infinite system, but the property considered for it does not match the limit of the property as defined on the finite systems. In more detail, we have:

(2): *There is no limit system*: There is no infinite system; though there is, or may be, a limit of the property of the finite systems. In the notation just introduced: there is no $\sigma(\infty)$, but there is, or may be, a limit, $\lim v(f(N))$ i.e. $\lim v(f(N), s_N)$. (So again, the property is usually given as the value of a quantity for an appropriate sequence of states).

(3): *Limit property and limit system disagree*: There is an infinite system $\sigma(\infty)$; (again, usually obtained as a limit of the state spaces of finite systems). But although the property of the finite systems (again: usually the value of a quantity) tend to a limit, this limit is *not* the value of the corresponding quantity on $\sigma(\infty)$. In our notation: $\lim v(f(N)) \neq v(f(\sigma(\infty)))$.

Note that each case is defined in terms of a single property or quantity f ; or more exactly, in terms of a single family $f(N)$, with maybe also $f(\sigma(\infty))$ (with a sequence of states understood). But of course, a typical theory considers many quantities, and so can illustrate more than one case. This happens in Norton's examples; and, as we will see in Section 2.1.C, it happens in ours.

Norton also proposes that his trichotomy prompts a useful regimentation of the words 'idealization' and 'approximation', whose usage varies widely (his Section 2, pp. 208-211). Thus in case (1), Norton says that the infinite system is an *idealization*. The point of an idealization is that its property (i.e. $v(f(\sigma(\infty)))$, the value of the quantity on the infinite system) gives an inexact description of what Norton calls the 'target system'—i.e. the systems $\sigma(N)$ for realistic i.e. finite N . (Of course, N might be very large, as in thermodynamics—cf. the kettle at the end of Section 1.) Furthermore, the numerical agreement in the limit means that, although the description is inexact, it is accurate *enough*. (The inexactness is just the difference between the limit of a sequence $\{v_N\}_N$ of values, and a member v_N of the sequence with a suitably large N to be close enough to the limit.) And typically, its small inaccuracies are justified by it being much more tractable for calculation and manipulation than the finite- N descriptions; (cf. Butterfield 2011, Section 3.3, p. 1076-1082).

On the other hand, Norton says that in case (2) there is a (*good*) *approximation*. That is: the limit $\lim v(f(N))$ gives a description of the target system that does not come from an idealization, and is inexact—but accurate *enough*, at least for high enough N . Finally, Norton says that in case (3), we have a *poor approximation*, because the property of the infinite system is not accurate enough a description of the target system.⁵

⁵Norton goes on to classify various limits, especially in thermal physics, e.g. the thermodynamic, continuum and Boltzmann-Grad limits, in terms of his cases. His main point is to argue that

2.1.C Comparison As we have seen, Norton is not concerned to define ‘emergence’, or indeed reduction: unlike Butterfield. But it is natural to suggest that Norton’s case (1) (‘idealization’) corresponds to Butterfield’s reconciliation of emergence with reduction. That is: the limit system’s limit property is the novel, so emergent, property; while its being the limit of the properties of the finite- N systems amounts to its being reduced.

In reply to this suggestion, we agree that a Nortonian case (1) might well be an example of emergence with reduction. But to say that his case (1) ‘corresponds’ to emergence with reduction is too strong; for two reasons.

(1): There could be novel and emergent properties without their being properties of a limit system. There might be *no* limit system, as in Norton’s case (2); so that the properties, despite their novelty and even importance, are uninstantiated. Besides, such a case could count as a reduction, owing to the novel property being a limit of the finite- N properties. Thus we see that Butterfield’s notion of emergence with reduction is in general logically weaker than Norton’s case (1). No worries, say we: it reflects the flexibility of the idea of emergence.

(2): We should of course allow reduction to use properties other than the finite- N version of the limit property. Recall our remark just after Norton’s case (3): viz. that a theory considers many properties (quantities). Thus there might well be examples in which a limit property is reduced using finite- N properties, i.e. quantities $q_1(N), q_2(N), \dots$, that are quite different from the finite cousin $f(N)$ of the limit property. (In this way, Butterfield’s notion of reduction as deduction is more flexible than Norton’s trichotomy, with its focus on a single sequence of properties: though of course, there is no conflict between our respective doctrines.) This second reason prompts another general comparison: as follows.

(3): It is natural to suggest that Norton’s case (3) (‘poor approximation’) corresponds to a failure of reduction, at least in Butterfield’s proposed sense, viz. as deduction. But again to say ‘corresponds’ is too strong, because of what we just said in (2). That is: there could be a deduction of the limit property, using quantities $q_1(N), q_2(N), \dots$, that are quite different from its finite- N cousins $f(N)$. Besides, this limit property could be important and even novel, and so emergent; even though it does not “mesh” with its finite- N cousins $f(N)$.

So much by way of general comparison. In the rest of this paper, we will see two main examples of the scenarios of emergence with reduction, just envisaged in (1) to (3) above.

The first example corresponds to the first reason (1) above. It concerns QCD and SYM equally; and besides, several other quantum field theories. Even for a finite number N of colours, the question whether an interacting quantum field theory rigorously exists is a subtle issue: which we will address in Section 2.2.1. But (1)

renormalization group methods are approximations, not idealizations; (his Sections 4.3, 5.2-3, pp. 219-223, 225-227). But we will not here try to systematically compare the ‘t Hooft limit with limits in statistical mechanics.

above concerns, rather, whether the limit system—i.e. for us, a gauge theory with an infinite number of colours—rigorously exists. Happily, we will report in Section 2.2.1 indications that for QCD and SYM, the limit theory does exist. But even if they do *not* exist, some of our three properties ((1) to (3) of Section 1) will exhibit emergence with reduction in our sense—as envisaged in (1) above. We will also suggest that this verdict accords with how theoretical physicists talk and write about the ’t Hooft limit (e.g. Polyakov (1987, Chapter 8.1), Gross 1999, p. 588).⁶

The second example corresponds to the second and third points, (2) and (3) above. We will see that SYM has been shown to have our third property, integrability, at the limit $N = \infty$, with a vivid representation using spin chains, by arguments that use—*not* integrability at finite N —but a different property: viz. the finite versions of our second property, planarity. That is: the arguments use the increasing dominance of Feynman diagrams that can be drawn on a plane. But despite using a different property, integrability is deduced, yet novel: emergence with reduction, again.

So far, this comparison has highlighted differences between Norton’s discussion and our previous one. We should also note a similarity. Both Norton and Butterfield, like most previous philosophical discussions, assume that the finite systems $\sigma(N)$ exist; and rightly so, for the examples they consider. But again: this involves two differences from our present study of the ’t Hooft limit.

(a): First: our parameter N , running through the positive integers, is not the number of degrees of freedom, since that is infinite in a field theory. Nor is it a kindred notion like the number of components, as in Norton’s examples of the number of squares in a lattice, or the number of atoms in a sample. As we said in Section 1, our N will be the number of colours: the N th theory describes its gauge field by an $N \times N$ matrix field, and so has gauge group $SU(N)$, rather than QCD’s $SU(3)$.⁷

(b): Second: since N labels interacting quantum field theories, the question whether even the finite N theory rigorously exists will need to be addressed: cf. the next Section.

2.2 The schema and trichotomy in QCD and SYM

We turn to summarizing how what follows relates to Norton’s trichotomy and Butterfield’s reconciliation of reduction and emergence. In short: we will classify in terms of Norton’s trichotomy, Section 1’s three properties that a quantum field theory—in an appropriate sequence of such theories, labelled by N —can have. The properties are, in short: (1) the high-energy behaviour of interaction strengths; (2) planar Feynman

⁶Later, we will briefly consider the suggestion that the limit theory is indispensable for understanding the finite- N case. We note here that even if this is true, it does not mean that infinitely many colours of the limit theory are physically real. Rather: if it is true, then we get a good understanding of a world, which is well but inexactly described by a heuristic theory with three colours i.e. QCD, by exploiting a cousin limiting theory.

⁷Since for each N , the $N \times N$ matrix has $N^2 - 1$ independent components (since the group is $SU(N)$), there are $N^2 - 1$ gauge fields.

diagrams, and ensuing string-like structures; and (3) being integrable. We will see that in all cases, there is good limiting behaviour: and in most cases, Norton’s case (1), idealization.

We will need a preliminary discussion about rigour (Section 2.2.1). Then in Section 2.2.2, the classification of the properties will be straightforward.

2.2.1 Do gauge theories exist? Should we dive in?

We admit at the outset that, so far as is known, the theories we will consider, QCD and maximally supersymmetric Yang-Mills (SYM), are not defined, even at finite N , as precisely as a mathematician would require. For even at finite N , these are interacting quantum field theories in 3+1 dimensions, whose rigorous existence remains to be proven. So as we mentioned, there is a contrast with philosophical discussions of statistical mechanics. There, finite N means finitely many degrees of freedom, and the theory or system is rigorously defined; even if its dynamics or other properties are intractable, and maybe easier to understand in the limit $N \rightarrow \infty$.

In the foundations and philosophy of quantum field theory, this is a familiar predicament: the tension between the often conflicting virtues of rigour, as in constructive or algebraic quantum field theory (AQFT), and heuristic power, as in functional (path-integral) quantum field theory. Physicists recognize this tension; although broadly speaking, the quantum field theory community is large and liberal enough to accommodate the two styles or *genres* of work, labelled ‘mathematical physics’ and ‘theoretical physics’.⁸

Philosophers of quantum field theory also recognize the tension. Some say philosophers should resist engagement with ‘merely heuristic’ formalisms: the time is not yet ripe for philosophical analysis. Others say that philosophers should “dive in”. A recent example of the debate between these two stances, and how they mould one’s assessment of both theories and research programmes, is given by papers by Fraser (2011) and Wallace (2006, 2011).⁹ As is already evident, we are for diving in! We see three reasons for doing so.

The first reason is just that despite the lack of rigour in the present-day physics, we will be able to classify our three properties in terms of Norton’s philosophical

⁸Agreed, the two styles seem to enjoy a healthy sibling rivalry, rather than brotherly love! Cf. Jaffe and Quinn (1993), and Atiyah et al. (1994). For a glimpse of the tension, focussed on questions central to this paper, viz. whether QCD’s asymptotic freedom needs to be proved and is sufficient grounds for believing that QCD rigorously exists, we recommend the brief exchange between Jaffe and Gross in Cao (1999, pp. 164-165). Jaffe, the mathematical physicist, urges that proofs are needed. The theoretical physicist Gross agrees, but emphasizes that he already fully believes QCD rigorously exists: if it did not, a problem would surely have already been found. Cf. also Gross (1997, pp. 57-62, 1999, p. 571), ’t Hooft (1984), and the exchange between theoretical physicists at the end of the latter. For a substantial discussion of the existence of quantum field theories from the viewpoint of constructive field theory, cf. Rivasseau (1991, especially Parts I, III.5).

⁹For a recent general discussion of rigour in physics, cf. Davey (2003).

taxonomy: rigour will not be needed, in order to connect to these philosophical concerns.

The second reason takes longer to state. For it introduces the notions of asymptotic freedom and conformal invariance, which will be crucial in the sequel; and it also leads to (i) discussing whether the theories we are concerned with exist in the 't Hooft limit, and (ii) two further philosophical comments. So all this is in Section 2.2.1.A: which will also lead to our third reason for diving in (given in Section 2.2.1.B).

2.2.1.A: Asymptotic freedom and conformal invariance :

Some considerations about quantum field theory suggest that the gauge theories we will consider *do* exist. Indeed, there are two points here:

(A): a negative one, with a longer history, suggesting that some *other* quantum field theories do not exist; and

(B): a positive one, with a more recent history, that *our* theories probably do exist.

Broadly speaking, the contrast depends on the type of fields in the theory concerned: scalars and fermions are in general subject to (A), while non-abelian gauge fields enjoy (B); and theories that combine both types of field need to be analysed individually, case by case, to see whether the bad behaviour due to scalars and fermions is outweighed by gauge fields' good behaviour. We also take it that this contrast is uncontroversial; cf. e.g. Weinberg (1995, Section 18.3, pp. 130-139), Gross (1997, pp. 57-62; 1999, pp. 573-576), and for a historical introduction, Cao (1999, Sections 10.2-10.3, pp. 290-305). We will begin with the negative point, (A).

(A): *In favour of non-existence:*

For many years, various results have suggested that some interacting quantum field theories, such as quantum electrodynamics (QED), do not exist, owing to the coupling constant becoming infinite at finite energy. For example, Landau thought this sort of bad behaviour was generic in quantum field theories; (hence the jargon that the finite energy at which the coupling blows up is called 'a Landau pole'). This suspicion has been confirmed for certain theories by rigorous non-existence results: Aizenman (1981) proved that an interacting scalar quantum field theory in more than four spacetime dimensions, with a non-zero ϕ^4 interacting term, has such behaviour. (For details, cf. Callaway (1988, Sections 1-3).)

Note that this behaviour, the coupling constant becoming infinite, is conceptually different from the time-development of an individual solution becoming singular: different, and arguably more dismaying, even if such singular solutions are generic, as for example, they notoriously are in general relativity. For this behaviour besets even the basic dynamical equations of the theory, i.e. its description of physics for arbitrarily short times. (This contrast between long and short times is of course crucial throughout physics: even in the classical mechanics of point particles, solutions can at long times become singular (Xia 1992). Cf. also Witten (1999, p. 1119; 2003, p.

27).)

We should here note one widespread response to this bad behaviour. Namely: we should take the theory to be “just” phenomenological, describing accurately a specific (low enough) energy-scale. Generalizing this response to other theories, for higher and higher energy-scales, amounts to the effective field theory programme: i.e. the suggestion that all our theories are “just” phenomenological—each describing accurately (or at least: accurately enough) the physics at some finite range of energies, but each going wrong, or even ill-defined, in its description of higher energies.¹⁰

(B): *In favour of existence:*

But (A)’s negative results concern quantum field theories that are neither *asymptotically free* nor *conformally invariant*. By briefly explaining these crucial properties, it will become clear that a theory with either property might well rigorously exist; and we will see later that our theories enjoy one or other property. In fact: QCD has the first, and SYM the second. Both properties arise from the Wilsonian perspective on renormalization; i.e. the renormalization group flow systematically redefining coupling constants as the energy-scale varies. We will first discuss these matters for finite N ; and then turn to $N = \infty$, i.e. whether there rigorously exist theories QCD(∞) and SYM(∞). This will echo Section 2.1’s discussion of the schema and trichotomy.

Asymptotic freedom means that as the energy-scale tends to infinity, the coupling constant tends to zero: the particles’ interactions die away—they do not feel each other. In terms of the renormalization group flow: the beta-function, which controls how the coupling constant varies with energy-scale, is negative and so drives the coupling constant, and thus itself, to vanish in the limit of infinite energy-scale. In the jargon: the theory has an ultra-violet Gaussian fixed point; where ‘ultra-violet’ means high-energy, and ‘Gaussian’ means free, i.e. no interactions. Obviously, this is a very striking (Nobel-winning!) feature of a theory. It radically simplifies the physics at higher and higher energies: in exactly the regime where we usually fear our quantum field theory will break down, the theory becomes free. Thus asymptotically free theories escape the above bad behaviour, and the Landau-pole arguments that they do not rigorously exist.

Conformal invariance means for most theories, including ours, that the theory’s beta-function is zero at all energy-scales.¹¹ Thus there is a fixed point in the degenerate sense that the coupling constant does not “run”, i.e. does not vary with energy. Thus the bad behaviour associated with a Landau pole is avoided. We should admit that (unlike QCD’s being asymptotically free), particle physics is, so far as we know,

¹⁰Hartmann (2001) is a good philosophical discussion of how such theories can have epistemic virtues, even without having a rigorous mathematical formulation. We should also admit that despite what we have said, some distinguished mathematicians are working to make sense of theories in case (A): e.g. Connes (2003).

¹¹This last clause really means scale-invariance. But for most theories, this implies conformal invariance; and throughout this paper, the difference will not matter.

not conformally invariant. But this of course does not prevent conformal invariance from being conceptually important: in particular, by being sufficient for the theory’s rigorous mathematical existence—as we urge here. Indeed, there are rigorous formulations of some conformally invariant theories in two dimensions; (Segal (2004); for an introduction, cf. Gannon (2008)).¹²

Thus asymptotically free or conformally invariant theories prompt a positive or optimistic outlook: such theories probably do rigorously exist. This outlook is well-nigh universal in theoretical physics: for example, recall the views of Gross and ‘t Hooft about asymptotic freedom, reported in footnote 8. Besides, as mentioned in Section 1: there are some supersymmetric theories whose beta-function has been computed exactly, and some of these are indeed asymptotically free or conformally invariant (Shifman 2012, pp. 531-533).

Besides, there are various general results relating asymptotic freedom to other important features of a theory: so these results are conceptually important. For example: the Coleman-Gross theorem says—roughly speaking!—that a necessary condition for asymptotic freedom is the use of non-abelian gauge fields. We will glimpse this for QCD in eq. 3.4 (in Section 3.1): for $SU(N)$, the gauge fields contribute a term $\frac{-11}{3}N$ to the beta-function, so that, unless outweighed by other terms dependent on the fermions, the function is negative. (For more details, cf. Gross (1999, p. 576-577), Zinn-Justin (2002, Chapter 32).) This is certainly one of the main reasons why gauge fields are important: if nature “wants to be well-defined” by being asymptotically free, then she must make use of them!

So much by way of generalities. We turn to our theories, QCD and SYM. For QCD, asymptotic freedom holds at all finite integers N . Usually, this is only stated for the physical value $N = 3$; just as we ourselves stated above. But it is easy to argue that it holds for all N . We will see in Section 3.1 (again, from eq. 3.4) that as we let N go to infinity in QCD, asymptotic freedom still holds for each value of N . Similarly for the conformal invariance of SYM. We will see in Section 4 (eq. 4.3) that as we let N go to infinity in SYM, conformal invariance still holds for each value of N .

The $N = \infty$ cases:

Of course, given the distinction between *at* the limit, and *on the way* to it—more specifically, Butterfield’s schema and Norton’s trichotomy in Section 2.1—we should ask whether our theories rigorously exist *at* the ‘t Hooft limit. As we discussed, the schema allows emergent properties at the limit without requiring the existence of the theory *at* $N = \infty$. On the other hand, Norton’s trichotomy, by definition, separated the case (1), where the theory exists, from case (2) where it does not. So in order to classify our three properties, (1)-(3) of Section 1, in Norton’s trichotomy, we need to

¹²Conformal field theories are also conceptually important for other reasons: they appear in the world-sheet formulation of string theory, in the AdS/CFT correspondence, and in the description of second-order phase transitions (e.g. Cardy 2008).

address this question, whether the theories exist at the limit.

As you would expect, this is at present less well-understood than the finite- N situation. Nevertheless, we think there are two general reasons to believe they do exist: indeed, reasons that also apply more generally to other quantum field theories.

(i): First: it is expected that in the 't Hooft limit, some of these theories become classical, in the sense that their path-integral reduces to the saddle-point approximation; (e.g. Polyakov (1987, Chapter 8.1), Gopakumar and Gross (1994, Section 1)). And as we already announced: for SYM, this classicality even gives integrability and a correspondence to spin chains (cf. Section 5.2). This integrability is important evidence that SYM rigorously exists in the 't Hooft limit.¹³

(ii): Second: there is some pure mathematical understanding of the gauge structure of these theories at the 't Hooft limit, especially the Lie algebra $\mathfrak{su}(\infty)$ of their group $SU(\infty)$. Cf. Hoppe (1989); or for a review, Rankin (1991, Chapter 2).

Two philosophical comments:

This discussion prompts two further philosophical comments. First: It suggests a compromise in the debate among philosophers about whether to dive in to assessing heuristic quantum field theory. Namely: there are different cases, i.e. theories, and one should not have a uniform stance for all of them. For theories that probably do not exist, as in (A), we should treat cautiously their apparent ontological claims, or 'world-picture'. So their assessment is likely to be a matter of epistemology, not ontology; especially about assessing the effective field theory programme mentioned at the end of (A); (cf. footnote 10). But for theories that probably do exist (like QCD and SYM!), philosophically assessing their ontology or world-picture is of course worthwhile. And this is even true for a theory like SYM that probably does not represent nature, if it helps us understand those that do—as SYM is believed to, cf. Section 1.¹⁴

In saying this, we do not mean to favour for philosophical discussion the asymptotically free and conformally invariant theories, i.e. to favour case (B) over case (A). One of the main virtues of the theories in (A) is that they are tractable at the low energy-scales we can observe, i.e. perturbation theory is reliable. And correspondingly, the down-side for asymptotic freedom in case (B) is the vice called 'infra-red slavery': i.e. at low energy-scales, the theory is very difficult to calculate perturba-

¹³Some other theories are also known to be integrable in the 't Hooft limit; the Gross-Neveu model, a theory of fermions in 1+1 dimensions, becomes equivalent to an integrable classical sigma-model (Gross 1999 p. 585-593).

¹⁴This last comment is not meant to suggest that we believe that on the other hand, QCD truly represents nature at arbitrarily high energies. Of course, we expect that it does not, owing to quantum gravity effects at Planck-scale energies. More generally, we endorse the consensus that ontology, more generally philosophical interpretation, is worthwhile even for physical theories known not be true in all details and-or in all regimes: a happy consensus, since otherwise philosophers of physics would have no work to do. For discussion, cf. e.g. Van Fraassen (1991, Chapter 1) and Belot (1998, Section 5).

tively, due to the strong coupling. So one needs non-perturbative techniques—and the 't Hooft limit is one of these.

Second: The contrast between (A) and (B) prompts a warning about efforts to mathematically understand the usual Higgs mechanism for spontaneous symmetry breaking within rigorous quantum field theory: an effort enjoined by some of the last decade's philosophical literature (e.g. Earman (2003, Section 3f., p. 338f.; 2004, Section 6f., p. 183f.), Healey (2007, p. 172)). Namely: although the pure gauge electroweak theory with gauge group $SU(2) \times U(1)$ is asymptotically free, the theory with the Higgs boson (a scalar!) added in is probably *not* asymptotically free: and so probably does not rigorously exist; (cf. Weinberg 1995, pp. 153-154).

2.2.1.B: Some meanings of 'existence' :

Our third reason for diving in arises from our second reason; but it is broader. Namely: the spectrum of degrees of rigour that physics exhibits is itself grist to the mill of philosophy! Thus one can envisage different precise meanings that a physicist intends by saying that a theory 'exists' or 'is well-defined'; and with these meanings in hand, a survey classifying which theories are presently known to enjoy which of the meanings would be philosophically very illuminating. We will not try to formulate such a spectrum of meanings and survey of theories. But we will sketch how the discussion so far gives the materials to do so. (In the next Section, it will become clear how this bears on the classification of our theories' three properties.)

First, it is clear enough what we have intended by the phrase 'rigorous existence': existence according to the standards prevalent in axiomatic/algebraic or constructive quantum field theory. A bit more exactly: this means the theory's having a consistent model in the sense of the work of Haag and Wightman, and their collaborators.

The other weaker meanings of 'existence' or 'well-definedness' are versions of the idea that the quantum field theory has finite behaviour at all energy-scales, especially at arbitrarily high energies. We have seen two important ways to make this precise: asymptotic freedom and conformal invariance. We should mention a third: *asymptotic safety*. This means that as the energy-scale tends to infinity, the coupling constant tends to a non-zero value. So this notion combines there being a renormalization group flow, as in asymptotic freedom, with the limit, i.e. the fixed point, not being Gaussian/free, as in conformal invariance. Cf. Weinberg (1979, pp. 798-809; 1997, p. 249), who suggests that this notion could apply to quantized gravity.

2.2.1.C: Conclusion :

So much by way of stating our reasons for enthusiastically diving in: the waters of heuristic quantum field theory are exhilarating, rather than murky! But finally, we should confess that it is not only the *existence* of our theories, QCD and SYM, that remains to be rigorously proved, even for finite N . Also: various results about them,

which we will take in our stride, e.g. that SYM is conformally invariant, and the recent results on integrability at infinite N that Section 5 emphasizes, are *not* proven as rigorously as one would hope for—and as in, for example, theorems in AQFT. For example, these results are usually obtained by computations in an expansion.

Of course, taking such heuristic results in our stride does not mean we fully believe every such result can, or will one day, be rigorously proven. Thus to take the case of integrability: we already admitted that Section 6 will consider integrability failing at finite N . And more generally, we of course concede that integrability of a finite N interacting quantum field theory in 3+1 dimensions is surely rare indeed. Rather, our taking such results in our stride means two things.

(i): The results are secure enough that, we say, the time is right (or ripe enough!) for philosophical assessment.

(ii): Returning to the question whether our theories, QCD and SYM, exist: we shall henceforth assume that they *do*—even in the ‘t Hooft limit. Of course, we are not sure. But they probably do: and it would be cumbersome to continually qualify our statements by repeating that they might not. Onward!

2.2.2 How the properties will be classified

So we henceforward assume that QCD and SYM exist, even in the ‘t Hooft limit. Now suppose that taking the limit preserves some property of the finite N theories, in the sense of Norton’s case (1), idealization. That is: the limit of the properties of the finite N theories is the property of the limit theory. We shall claim that this happy “meshing” case holds good for both QCD and SYM, as regards our first two properties, i.e. the behaviour of the beta-function and planarity. And planarity illustrates Butterfield’s emergence with reduction; (cf. Section 2.1.C).¹⁵

Our third property, integrability, fares a little differently. It also emerges in the $N = \infty$ limit; and with a vivid representation using spin chains. But so far as is known, this does *not* follow from the finite- N theories also being integrable: they may well not be. Instead, it follows, broadly speaking, from these finite- N theories becoming increasingly planar, as N tends to infinity. So in Norton’s trichotomy, integrability illustrates case (3), which he labels ‘poor approximation’. But as stressed in Section 2.1.C, such an example can instantiate Butterfield’s emergence with reduction: just because reduction can use properties (especially, quantities) of the finite- N

¹⁵One last *cri de coeur* from our over-cautious *alter ego*! Suppose that the theories do not rigorously exist in the ‘t Hooft limit, but are “merely” asymptotically free/conformally invariant there. Indeed, we can cautiously suppose that even at finite N , they are merely asymptotically free/conformally invariant. Nevertheless, we shall see that our first two properties for the finite- N theories have a limit as N tends to infinity. (That is: ‘have a limit’, by the standards of the heuristic study of asymptotically free/conformally invariant theories.) So clearly, this is Norton’s case (2), good approximation. That is: one can use the limit of the property as an inexact, but accurate enough, description of those theories in the sequence that have a high enough value of N . And again, planarity illustrates Butterfield’s emergence with reduction; cf. (1) of Section 2.1.C.

theories that are not the finite cousins of the reduced limit property. We shall claim that integrability in the 't Hooft limit is indeed emergent but reduced: so that here, Norton's label, 'poor approximation', is unfortunate, since it connotes failure where in fact there is a vivid success.

So we now close this Section's long preview of what is to come, by spelling out the last two paragraphs; in three comments, (a)-(c).

(a): Since QCD rigorously exists in the 't Hooft limit, Section 3 will illustrate Norton's case (1), i.e. idealization, with two important properties of these theories: the beta-function being negative, thus securing asymptotic freedom (Section 3.1); and planarity (Section 3.2). As mentioned in Section 1, planarity will be associated with novelty: the appearance of string-like structures and a classical regime—thus illustrating emergence combined with reduction.

(b): When we turn to the supersymmetric theory SYM (Sections 4 and 5), the classifications of these two properties will be as in (a); (with a minor qualification which is clear from (B) of Section 2.2.1.A). That is: the classifications are as for QCD and its higher N "cousins", despite the theory being very different. Again, assuming rigorous existence: we have Norton's case (1), idealization, and for planarity, Butterfield's emergence with reduction. The minor qualification is just that the property of the beta-function to consider is now, not it being negative, but it being zero. For SYM is conformally invariant, which means that its beta-function is zero at all energies. So there is no scope or need for argument that it is driven to zero at high energies.

(c): In Section 5, we will address, for SYM, the property of integrability; and as announced, we will see that it obtains in the limit. Indeed, this integrability is an important part of our evidence that SYM rigorously exists in the 't Hooft limit. This will again illustrate Butterfield's emergence with reduction. But the classification of integrability in Norton's trichotomy will be: case (3), rather than case (1), as just reported in (a) and (b) for the beta-function and planarity.

The reason is that it is much easier to ascertain at finite N these first two properties than integrability. The property labelled 'the beta-function' is really a matter of that function being negative or zero (the signatures of asymptotic freedom or conformal invariance). And one can ascertain at finite N , at least perturbatively, that the function is negative or zero. Similarly for planarity. We will see in Section 3.2 that it means the numerical dominance of those Feynman diagrams that can be drawn on a plane without intersecting lines. Again, one can ascertain at finite N the contribution to a given process made by such diagrams: the point being that for any process, as N grows, these diagrams' contribution swamps all other contributions. But we will see in Section 6 that it *is* difficult to ascertain at finite N integrability in Section 5's sense. The reason is that this sense is relative to a perturbative expansion in the parameter λ , the 't Hooft coupling: $\lambda := g^2 N$. That is: at each order in λ , the system is integrable: to be precise, the scaling dimension is given by the eigenvalues of a spin chain's Hamiltonian. But it is difficult to carry this approach over to the case of finite

N .

Finally, this classification of integrability prompts a broader suggestion. Namely: perhaps it will be a theory (or even a fragment of a theory) at *infinite* N , but not at finite N , that is computationally tractable—and that we can hope to exploit to better understand the finite N case. We will briefly return to this in Section 7. This is not to suggest that the physics community regards focussing on the infinite N theory as the unique, or best, approach to taming interacting quantum field theories: of course, there are several other approaches.

3 The ‘t Hooft limit in QCD

In this Section, we sketch: (i) why one seeks an approximation scheme in QCD (Section 3.1); and (ii) how ‘t Hooft’s choice of the number $N \equiv N_c$ of colours as the parameter of an expansion leads, in the limit of large N , to planar diagrams, i.e. diagrams without intersecting lines (Section 3.2).

3.1 The need for an approximation scheme

QCD is a rich and complicated theory. Several of its essential features are still poorly understood, within physics—let alone philosophy! These include confinement, dynamical mass generation and chiral symmetry breaking. Besides, confinement means that in the low energy regime, QCD becomes strongly coupled; and accordingly, increasingly difficult to calculate. As mentioned at the end of Section 2.2.1.A, this infra-red slavery is the down-side of the fact which we will later focus on: that QCD is asymptotically free, i.e. that the coupling constant decreases as the energy increases. For QCD, the best available approach to deal with the calculational difficulties is to use numerical simulation on a lattice. (Incidentally, here we connect with Norton’s examples of lattices and real space renormalization; cf. Section 2.1.B.)

But even apart from calculating specific problems, QCD is so complicated a theory that we cannot expect to obtain exact solutions (Dolan et al. 2003). Therefore, even apart from using lattices, we need to find some sort or sorts of approximation scheme.¹⁶ Since a good approximation scheme is traditionally considered to require—and is probably only possible if there is—an appropriate expansion parameter, we face the question: what possible expansion parameter does QCD contain?

The ordinary coupling constant is not really a free parameter in QCD. (Though we will not need the details, the reason is that as a result of the renormalisation group flow, the coupling constant is absorbed into defining the scale of masses by dimensional transmutation.) Indeed, this is one of the most important facts we know about QCD: not least because it is this fact that makes the theory both difficult to

¹⁶Here we mean ‘approximation scheme’ in physics’ usual wide sense, not Norton’s specific sense of Section 2.1.

calculate and hard to understand. Thus the theory has no obvious free parameter that could be used as an expansion parameter: it is apparently a theory without parameters. Thus one must hope to find a non-obvious free parameter.

Famously, 't Hooft (1974) pointed out that QCD has a non-obvious candidate for an expansion parameter. He suggested that one should generalize QCD, from three colours and an $SU(3)$ gauge group, to $N \equiv N_c$ colours and a $SU(N)$ gauge group.¹⁷ More precisely, he expanded the partition function and the correlation functions of a $SU(N)$ gauge theory in powers of N ; and argued that the theory simplifies when the number N of colours is large. In graphic terms: complicated Feynman diagrams at finite N are replaced by much simpler *planar* diagrams, and so one has to cope with far fewer diagrams—as described in Section 3.2. This has proven very useful in lattice QCD (cf. Teper (2009) for a recent review); and it prompted the hope that one could solve the theory exactly at $N = \infty$, and then one could better understand QCD itself by doing an expansion in $1/N = 1/3$. Although these hopes have not yet come true, the results to be surveyed in the rest of this paper are surely progress.

We turn to some details. 't Hooft considers a $SU(N)$ gauge theory, with N_F flavors of fermions (for short: quarks), transforming in the fundamental representation of $SU(N)$. There are $N \equiv N_c$ colours; the gauge bosons (for short: gluons) transform in the adjoint representation of $SU(N)$ (dimension = N). So QCD is the special case: $N = 3, N_F = 3$. The gluon field is an $N \times N$ traceless hermitian matrix; it is written in terms of T^A , the generators of $SU(N)$, as $A_\mu = A_\mu^A T^A$. Here, μ is a spacetime index, $\mu = 0, 1, 2, 3$; and the superscript A is an index in the adjoint representation, $A = 1, 2, \dots, N$. The element of A_μ on the a th row and b th column ($a, b = 1, \dots, N$) is written $(A_\mu)_b^a$. The matrices T^A are normalized so that $\text{Tr} T^A T^B = \frac{1}{2} \delta^{AB}$, and the covariant derivative is:

$$D_\mu = \partial_\mu + i \frac{g}{\sqrt{N}} A_\mu. \quad (3.1)$$

The coupling constant has been chosen to be g/\sqrt{N} , rather than g , because this will lead to a theory with a sensible (and non-trivial) large N limit. The field strength is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i \frac{g}{\sqrt{N}} [A_\mu, A_\nu],$$

and the Lagrangian is, with the usual Dirac gamma-matrices and fermions $\psi_k, k = 1, \dots, N_F$:

$$L = -\frac{1}{2} \text{Tr} F_{\mu\nu} F^{\mu\nu} + \sum_{k=1}^{N_F} \bar{\psi}_k (i \gamma^\mu D_\mu - m_k) \psi_k. \quad (3.2)$$

¹⁷For an introduction, we recommend Witten (1979, 1980) and Coleman (1985, Chapter 8). Witten (1980) includes a motivating discussion of taking the limit of elementary wave mechanics (atomic physics) in N spatial dimensions, rather than the usual $N = 3$; and seeing the theory simplify at $N = \infty$. 't Hooft's proposal had precedents in 1960s work in statistical mechanics; cf. Brezin and Wadia (1993, Chapter 1).

The large N limit will be taken with the number of flavors N_F fixed. (It is also possible to consider other limits, such as $N \rightarrow \infty$ with N_F/N fixed.)

One way to understand the g/\sqrt{N} scaling of the coupling constant, and some features in the large N limit, is to look at the one-loop β -function. For most field theories, computing the β -function that defines the renormalization group flow is (as Section 2 indicated) very difficult. But remarkably, the one-loop contribution (first quantum correction) gives a lot of information about the theory's ultra-violet behaviour and so its existence (Gross 1999, p. 571).

For our $SU(N)$ gauge theory, the one-loop β -function β_1 , written in terms of the coupling constant g and energy-scale μ , is:

$$\beta_1(g) := \mu \frac{dg}{d\mu} = -b_1 \frac{g^3}{16\pi^2}, \quad b_1 = \frac{11}{3} C_2(G) - \frac{4}{3} N_F C(R), \quad (3.3)$$

where $C_2(G)$ is a numerical constant characteristic of the representation of the gauge group G in which the gluons transform, viz. a quadratic Casimir invariant; and $C(R)$ is another Casimir invariant characteristic of the representation in which the fermions transform. For gluons in the adjoint representation, $C_2(G) = N$; and for fermions in the fundamental representation, $C(R) = 1/2$. With this understanding, Eq. (3.3) yields

$$\beta_1(g) = -\frac{g^3}{16\pi^2} \left(\frac{11}{3} N - \frac{2}{3} N_F \right). \quad (3.4)$$

The minus sign in Eq. (3.4) is the signal of that remarkable feature, asymptotic freedom (as long as the number of fermion flavors N_F is small enough), i.e. that g decreases at higher energies, with $\lim_{\mu \rightarrow \infty} g = 0$. We argued in Section 2.2.1 that this feature, asymptotic freedom, is a strong reason to believe that gauge theories exist.

We now turn to analysing the behaviour of gauge theories in the planar limit. Equation (3.4) does not have a sensible large N limit since the one-loop β -function, i.e. β_1 , is of order N . Replacing g by g/\sqrt{N} in eq. (3.4) (i.e. defining $g' := \sqrt{N}g$ and then writing g for g') gives

$$\beta_1 = \mu \frac{dg}{d\mu} = - \left(\frac{11}{3} - \frac{2}{3} \frac{N_F}{N} \right) \frac{g^3}{16\pi^2}. \quad (3.5)$$

The β -function equation now has a well-defined limit as $N \rightarrow \infty$. The N_F term is suppressed by $1/N$. Besides, the limiting formula, i.e. without $(2/3)N_F/N$, is the formula for the β -function for the pure-gluon sector of the theory. Thus the large N limit for QCD, with the coupling constant scaling like $1/\sqrt{N}$, is equivalent to taking the limit $N \rightarrow \infty$, and asymptotic freedom is preserved.

Returning to our project of classifying properties in terms of our schema and Norton's trichotomy (Sections 2.1 and 2.2.2): let us consider the property of having a negative β -function. For N sufficiently large, the first term in eq. 3.5 is negative

(indeed, we only need $N > \frac{2}{11}N_F$). So the situation is as we announced in (a) of Section 2.2.2. That is: (i) there is no ‘novelty’, and so no emergence, in the limit; yet (ii) since we are assuming that QCD rigorously exists in the ‘t Hooft limit, the negative β -function illustrates Norton’s case (1), i.e. idealization.

3.2 The emergence of planarity

To understand the theory in the $N \rightarrow \infty$ limit, we now analyse Feynman diagrams: we will see that they become planar. For this analysis, we need a simple way to count the powers of N in a given diagram. ’t Hooft noticed that this could be done with a new kind of graph with a line for each index a and b in $A_\mu \equiv (A_\mu)_b^a$. So double lines between vertices (rather than the usual one line) keep track of colour. The result is that every Feynman diagram in a perturbative expansion of the original theory can be written as a sum of ’t Hooft double line graphs: each double line graph gives a particular colour index contraction of the given Feynman diagram.

For our purposes, the main point about this double line notation is that one can think of each double line graph as a polyhedral surface obtained by gluing polygons together at the double lines. Since each line has an arrow on it, and double lines have oppositely directed arrows, one can only construct orientable polyhedra.¹⁸

To compute the N -dependence requires counting powers of N from sums over closed colour index loops, as well as factors of $1/\sqrt{N}$ from the explicit N dependence in the coupling constant. It is convenient to use a rescaled Lagrangian to simplify the derivation of the N -counting rules. We define rescaled gauge fields $gA/\sqrt{N} \rightarrow \hat{A}$, so that the covariant derivative is $D_\mu = \partial_\mu + i\hat{A}_\mu$; and rescaled fermion fields $\psi \rightarrow \sqrt{N}\hat{\psi}$. Then the resulting rescaled Lagrangian from Eq. 3.2 has an overall factor N in front. From this Lagrangian, one can read off the powers of N in any Feynman diagram.

Every vertex gives a factor of N , and every quark or gluon propagator gives a factor of $1/N$. In addition, every colour index loop gives a factor of N , since it represents a sum over N colours. But now we note that in the double line notation where Feynman diagrams correspond to polygons glued together to form polyhedra, each colour index loop is the perimeter of a polygon, and so corresponds to a face of the polyhedron. Thus one finds that a connected vacuum diagram (i.e. with no external lines) is of order

$$N^{V-E+F} =: N^\chi, \quad (3.6)$$

where V is the number of vertices, E is the number of edges, and F is the number of faces. But $\chi := V - E + F$ is a topological invariant, the famous Euler character; and for a connected orientable surface

$$\chi = 2 - 2h - b, \quad (3.7)$$

¹⁸This last remark follows from the theory’s using $SU(N)$. For a theory with $SO(N)$, the fundamental representation is a real representation, and the lines in a ’t Hooft graph would not have arrows; so that in this case, it is possible to construct non-orientable surfaces such as (polyhedral approximations to) Klein bottles.

where h is the number of handles, and b is the number of boundaries; (where handles and boundaries are themselves topological invariants). For a sphere and its homeomorphs: $h = 0$, $b = 0$, and so $\chi = 2$; for a torus and its homeomorphs, $h = 1$, $b = 0$, and so $\chi = 0$.

A quark is represented by a single line, and so a closed quark loop is a boundary. Thus every closed quark loop brings a $1/N$ suppression. Besides, the maximum power of N is two, from graphs with $h = b = 0$.¹⁹ These are connected graphs with no closed quark loops, and with the topology of a sphere. In a similar way, it can be shown that in the pure glue (no quarks) sector of the theory, the 't Hooft graphs with the highest power (viz. 2) of N , which will dominate as N grows, all have the topology of the sphere ($h = 0$).

Now we can see how planarity emerges in the $N \rightarrow \infty$ limit. Consider first the pure-gluon sector, and imagine removing one polygon from the “sphere”, so that one obtains a punctured sphere, with one puncture. This can be flattened into a diagram drawn on a flat sheet of paper, with the puncture as the outer perimeter. One can then glue back the removed polygon by thinking of it as all the paper exterior to the diagram. Thus the order N^2 graphs are planar diagrams: they can be drawn on the surface of a sheet of paper without having a gluon “jump” over another. That is, all points where gluon lines cross have to be interaction vertices.

A similar result, that planarity emerges for large N , holds good for diagrams that depend on quarks. The leading diagrams are of order N , with $h = 0$ and $b = 1$.²⁰ It turns out that these diagrams have the topology of a punctured sphere with one puncture, with one of the diagram’s quark loops forming the boundary of the puncture. One can then flatten out such a diagram into a planar diagram, as we did for gluons: the single quark loop forms the outer perimeter of the diagram.

To sum up: in the pure-gluon sector, the leading diagram has the topology of a sphere, but can be flattened onto the plane. And in the quark sector, the leading diagram is also planar, with a single quark loop forming its outer perimeter.²¹

3.2.1 Conceptual remarks

After these technicalities, we end this Section with four conceptual remarks: the first three about the physics, the fourth about our philosophical project of classifying properties.

(1): First: Planarity in the limit may at first seem a “merely technical” property, compared with e.g. asymptotic freedom. But not so, for two reasons.

¹⁹This corresponds to a vacuum energy of order N^2 : to be expected since there are $\mathcal{O}(N^2)$ gluon degrees of freedom.

²⁰One might expect the quark contribution to the vacuum energy to be of order N , since there are N quarks of each flavour: cf. the previous footnote.

²¹For simplicity, we have discussed only connected diagrams. One can obtain the N -dependence of a disconnected diagram by multiplying the N -dependences of all the connected pieces.

(i): It means that in the limit the perturbation expansion represents the theory completely, in the sense that the amplitudes for non-perturbative effects are suppressed. The heuristic reason is that such effects have amplitudes $\sim \exp(-1/g^2)$; but in the 't Hooft limit, we fix $\lambda := g^2 N$, and then we let N tend to infinity with λ fixed—so that $\exp(-1/g^2) \equiv \exp(-N/\lambda)$ tends to zero. This is not to say that the completeness of the perturbation expansion representation makes the theory easy to work with. One still has to sum over all planar Feynman diagrams (a problem attacked in the 1980s by authors such as 't Hooft and Rivasseau). Here lies the promise of SYM's integrability: for SYM, planarity leads very fortunately to tractable computation; cf. Sections 4 and 5.

(ii): The fact that the dominant diagrams at large N look like two-dimensional surfaces prompts the idea that these surfaces could be analysed as the propagation in time of a one-dimensional object, i.e. a string. Thus planarity, in addition to simplifying the theory, suggests a connection between quantum fields and strings. This leads to our second remark.

(2): Second: In the 1970s, similar string-like structures were noticed in other contexts. For example, in the context of lattice gauge theories, Wilson noticed that the strong coupling expansion involves a sum over two-dimensional surfaces, as a result of a propagation of one-dimensional objects, viz. (colour)-electric flux lines (often called 'Wilson loops').²² This led, naturally enough, to the suggestion that the string-like structures appearing in the different contexts were in fact the same; (though there was of course room for doubt, as stressed by Polyakov (2010)).

Polyakov also suggested that these string-like structures could involve an extra spatial dimension. His idea was that there was a crucial difference between the classical and quantum cases. Classically, the strings have only transverse oscillations. But after quantisation, they acquire an extra longitudinal mode, i.e. a Liouville mode; and this Liouville mode turns out to play the role of an extra spatial dimension. This idea became of more significance in later years, with the idea of an emergent spatial dimension in string theory, as in the AdS/CFT correspondence (Maldacena (1998)), mentioned in Section 1. We will return to this at the end of Section 5.

(3): Third: Another remarkable feature of the 't Hooft limit, is that gauge invariant observables \mathcal{O} become c-numbers. More precisely: the correlation function of \mathcal{O} at two spacetime points x and y (discussed in Section 1) factorizes in the 't Hooft limit:

$$\langle \mathcal{O}(x) \mathcal{O}(y) \rangle := \int \mathcal{D}A_\mu e^{-S[A_\mu]} \mathcal{O}(x) \mathcal{O}(y) = \langle \mathcal{O}(x) \rangle \langle \mathcal{O}(y) \rangle + O(1/N^2); \quad (3.8)$$

²²For an ontological viewpoint about these lines, cf. Healey (2007, Chapter 7). While we agree that any conceptual account of gauge theories must consider these lines and other extended objects, we must leave this task for another day.

(where for simplicity, we have written the path-integral definition only for the gauge-field A , and without normalization). This prompts an analogy with classical field theories, with correlation functions being characterised in the classical limit, $\hbar \rightarrow 0$, by $1/N^2$ in eq. 3.8 now playing the role of \hbar ; so that $\hbar \rightarrow 0$ corresponds to $N \rightarrow \infty$. (For a conceptual discussion, cf. Witten (1980, Section IV), Gopakumar and Gross (1995)).

(4): Fourth: Returning to our project of classifying properties (Sections 2.1 and 2.2.2): let us consider the property, ‘planarity’, that the theory’s Feynman diagrams that can be drawn on the plane give a dominant contribution to any process. Then the situation is similar to the end of Section 3.1. For N sufficiently large (how large depending on the process in question), the planar diagrams dominate; and in the limit $N \rightarrow \infty$ only planar diagrams contribute. The striking appearance of string-like structures and a classical regime certainly counts as novel, and so emergent, although reduced i.e. deduced by taking the limit. And since we are assuming that in the ‘t Hooft limit, QCD rigorously exists, we have Norton’s case (1), an idealization.

4 Introducing super Yang-Mills theory

4.1 Introducing the theory

We now ‘change horses’, i.e. consider a different theory than QCD and its higher- N cousins. Namely: $\mathcal{N} = 4$ maximally supersymmetric Yang-Mills theory, the study of which was initiated by Brink et al. (1977). Here, \mathcal{N} is the number of copies of the supersymmetry algebra: *not* the number of colours, N . However, the theory’s gauge group will be the familiar $SU(N)$. This theory is often called, for short, ‘ $\mathcal{N} = 4$ SYM’; where ‘SYM’ stands for ‘super Yang-Mills’. But we will not consider other values of \mathcal{N} ; so we will just write ‘SYM’.

We mentioned in Section 1 that this theory’s $N = \infty$ limit is simpler to study than that of QCD, and also exhibits planarity and integrability (details in Section 5). But there are also two other good reasons to study it.

First, it has various remarkable properties. Broadly speaking, it has a large amount of symmetry: which is the origin of the simplicity, planarity and integrability just mentioned. More specifically, it is conformally invariant, implying that it has no inherent scale. Classically, many theories are conformal, e.g. theories with only massless fields; (of course, Maxwell theory is the paradigm example). But SYM stays conformal even at the quantum level. In particular, its β -function is believed to be zero to all orders in perturbation theory; (cf. Section 4.2). And although QCD is not conformal, its being asymptotically free means that at high energies it is close to being conformal. Thus many essential features of high energy gluon scattering—which is very relevant for the LHC—can be analysed by studying gauge boson amplitudes in SYM.

Second, this theory is the gauge theory ‘side’ of the best-understood example of the gravity/gauge, or AdS/CFT, correspondence. (The gravity side is a certain string theory on a cousin of anti-de Sitter space: hence the label, with ‘CFT’ standing for ‘conformal field theory’—here SYM.) In Section 1, we postponed this topic to another paper. So suffice it to say here that since Maldacena (1998) introduced this correspondence, it has taken centre-stage in the study both of string theories and of high energy quantum field theories (including QCD). Besides, we will see at the end of Section 5 that the significance of its topic, integrability, lies largely in the light this sheds on AdS/CFT.

In this Section, we will first describe the fields that make up SYM, and sketch how they lead to a vanishing β -function and so conformal invariance (Section 4.2). Then we discuss the symmetries of SYM, and use them to define a class of operators, *primaries* (Section 4.3). In Section 4.4, we introduce the *single trace operators*: these function as ‘building-blocks’ of the gauge invariant operators, in the large N limit. We end the Section by introducing the relation to spin chains, which looks forward to Section 5’s discussion of integrability.

4.2 The vanishing β -function and conformal invariance

The fields contained in SYM are the gauge bosons A_μ , six massless real scalar fields ϕ^I , $I = 1 \dots 6$, four chiral fermions ψ_α^a and four anti-chiral fermions $\bar{\psi}_{\dot{\alpha}a}$, with $a = 1 \dots 4$. The indices $\alpha, \dot{\alpha} = 1, 2$ are the spinor indices of the two independent $SU(2)$ algebras that make up the four-dimensional Lorentz algebra. All fields transform in the adjoint representation of the gauge group, which is $SU(N)$; (unlike Section 3’s QCD, where the fermions transformed in the fundamental representation). The covariant derivative is defined, for χ one of the fields ϕ^I , ψ_α^a , etc. by:

$$\mathcal{D}_\mu \chi(x) := \partial_\mu \chi(x) - [\mathcal{A}_\mu(x), \chi(x)], \quad (4.1)$$

where \mathcal{A}_μ is obtained from A_μ by absorbing coupling constants. The corresponding field strength is then written $\mathcal{F}_{\mu\nu}$; and as usual, we have $[\mathcal{D}_\mu, \mathcal{D}_\nu] = -\mathcal{F}_{\mu\nu}$.

The fields that transform covariantly under the gauge group $SU(N)$ include: the scalars ϕ^I , the fermions ψ_α^a , $\bar{\psi}_{\dot{\alpha}a}$ and the field strengths $\mathcal{F}_{\mu\nu}$. Since these fields all live in the adjoint representation, their transformation under a gauge transformation is

$$\chi(x) \rightarrow \chi(x) + [\varepsilon(x), \chi(x)] \quad (4.2)$$

where $\chi(x)$ is the covariant field, and $\varepsilon(x)$ is the generator of the gauge transformation. We have explicitly included the space-time dependence of the fields to emphasize that this is a local transformation. By applying \mathcal{D}_μ to a covariant field $\chi(x)$, we can make other covariant fields $\mathcal{D}_\mu \chi(x)$, etc.; while the gauge connection $\mathcal{A}_\mu(x)$ does not transform covariantly.

We can use this listing of the field content to discuss the scaling of the theory: in particular, conformal invariance. We will now sketch why the one-loop β -function is

zero. For any $SU(N)$ gauge theory, the one-loop β -function β_1 is given by

$$\beta_1(g) := \mu \frac{\partial g}{\partial \mu} = -\frac{g^3}{16\pi^2} \left(\frac{11}{3} N - \frac{1}{6} \sum_i C_i - \frac{1}{3} \sum_j \tilde{C}_j \right), \quad (4.3)$$

where the first sum is over all real scalars with quadratic Casimir C_i and the second sum is over all Weyl fermions with quadratic Casimir \tilde{C}_j . Since all fields in SYM are in the adjoint representation, all the Casimirs are N . So mere arithmetic implies that, with six real scalars ($i = 1, \dots, 6$) and eight Weyl fermions ($j = 1, \dots, 8$), we have: $\beta_1(g) = 0$.

Going beyond one-loop, the β -function for SYM was shown to be zero for some higher loops; and accordingly, it is now believed that the β -function vanishes at all loops and maybe non-perturbatively, and hence (cf. footnote 11) that the theory is conformally invariant (Brink et al. 1983).

Here again we can summarize in terms of our classificatory project. In Section 3.1, we classified QCD's asymptotic freedom as a Nortonian case (1), since we assume that QCD rigorously exists in the 't Hooft limit. Similarly here, for SYM's property of conformal invariance. That is, as we announced in (b) of Section 2.2.2: the classification is the same: assuming that SYM rigorously exists in the 't Hooft limit, SYM's having a vanishing β -function is a Nortonian case (1).

4.3 Dilatation and primary operators

As just emphasised, the quantum SYM theory is conformally invariant, so that the Poincaré symmetry is extended to conformal symmetry in four dimensions, yielding a symmetry group $SO(4, 2) \simeq SU(2, 2)$. There is also the global symmetry group acting on the four copies of the supersymmetry algebra, the so-called R-symmetry: this group is $SU(4) \simeq SO(6)$. Putting these two together with the supersymmetry generators, we get as the overall symmetry group of SYM, the superconformal group in four dimensions: which is the graded Lie group $PSU(2, 2|4)$. The P stands for 'projective'.

Here we meet a crucial contrast with the situation for QCD. Classical QCD (with massless fermions in Eq.(2.1)) is conformally invariant (with the symmetry group: $SO(4, 2)$). But we have every reason to believe that when it is quantised, the conformal symmetry is broken, with only Poincaré symmetry remaining.²³ However for SYM, the big symmetry group $PSU(2, 2|4)$, including the classical conformal symmetry is unbroken by quantum corrections. This puts significant constraints on the quantised theory, and provides us with a powerful tool.

In this section we will briefly review some important aspects of the $PSU(2, 2|4)$ group, and how its structure, especially the bosonic subgroup $SU(2, 2) \times SU(4)$,

²³The general topic of classical symmetries being lost after quantization is called 'anomalies'.

implies the existence of operators of minimal scaling dimension, called *primary operators*, which will be crucial for our discussion of integrability. For simplicity, we will consider just the bosonic part of the symmetries.

The conformal group has fifteen generators. Ten generators belong to the Poincaré algebra, which has four generators P_μ of space-time translations, and six generators $M_{\mu\nu}$ of the $SO(3, 1) \equiv SU(2) \times SU(2)$ Lorentz transformations. The other generators of the conformal algebra are the four generators K_μ of special conformal transformations, and one generator D of dilatations. These generators satisfy the commutation relations

$$\begin{aligned} [D, P_\mu] &= -iP_\mu, \quad [D, M_{\mu\nu}] = 0, \quad [D, K_\mu] = +iK_\mu, \quad [P_\mu, K_\nu] = 2i(M_{\mu\nu} - \eta_{\mu\nu}D). \\ [M_{\mu\nu}, P_\lambda] &= -i(\eta_{\mu\lambda}P_\nu - \eta_{\lambda\nu}P_\mu), \quad [M_{\mu\nu}, K_\lambda] = -i(\eta_{\mu\lambda}K_\nu - \eta_{\lambda\nu}K_\mu). \end{aligned} \quad (4.4)$$

The dilatation operator D turns out to play a crucial role in the *quantum* structure of SYM. While the generators of the Poincaré subgroup of $PSU(2, 2|4)$ do not get quantum corrections, the dilatation operator D does:

$$D = D_0 + \delta D(g), \quad (4.5)$$

where D_0 is the classical operator and δD is the anomalous dilatation operator which depends on the coupling g .

Now let $\mathcal{O}(x)$ be a local operator in the field theory with scaling dimension Δ . (Recall Section 1's introduction to the idea of scaling dimension.) The physical idea of Δ is that it is the analogue of the mass in QCD. Technically: under the rescaling $x \rightarrow \lambda x$, the operator $\mathcal{O}(x)$ scales as $\mathcal{O}(x) \rightarrow \lambda^{-\Delta} \mathcal{O}(\lambda x)$; and the dilatation operator D is the generator of these scalings, by which we mean that $\mathcal{O}(x) \rightarrow \lambda^{-iD} \mathcal{O}(x) \lambda^{iD}$. The dimension Δ is $\Delta_0 + \gamma$; with Δ_0 the classical dimension corresponding to the classical operator D_0 in Eq.(4.5) and γ the anomalous dimension arising from quantum corrections corresponding to δD (Di Francesco et al. 1997). Thus to find the anomalous dimension γ of $\mathcal{O}(x)$, one considers its two-point correlator with itself:

$$\langle \mathcal{O}(x) \overline{\mathcal{O}}(y) \rangle \approx \frac{1}{|x - y|^{2\Delta}}. \quad (4.6)$$

In Section 5.1, we will give more details about computing anomalous dimensions; but we will now sketch how the action of D leads to the idea of primary operators.

The action of the dilation operator D on $\mathcal{O}(x)$ is

$$[D, \mathcal{O}(x)] = i \left(-\Delta + x \frac{\partial}{\partial x} \right) \mathcal{O}(x). \quad (4.7)$$

Now we apply D to $[K_\mu, \mathcal{O}(0)]$ and find, using the Jacobi identity, that

$$[D, [K_\mu, \mathcal{O}(0)]] = i[K_\mu, \mathcal{O}(0)] - i\Delta[K_\mu, \mathcal{O}(0)]. \quad (4.8)$$

Thus, the special conformal generator K_μ creates from $\mathcal{O}(x)$ a new local operator, $[K_\mu, \mathcal{O}(x)]$, which has dimension $\Delta - 1$. Aside from the identity operator, the local operators in a unitary quantum field theory must have positive dimension. Therefore, if we keep creating new lower-dimensional operators by commuting with the special conformal generators, we must eventually reach a barrier where we can go no further. Hence the last operator in this chain, $\tilde{\mathcal{O}}(x)$, must satisfy

$$[K_\mu, \tilde{\mathcal{O}}(x)] = 0. \quad (4.9)$$

Furthermore, it can be shown that for a given initial operator $\mathcal{O}(x)$, all the K_μ lead to the same barrier, i.e. the same $\tilde{\mathcal{O}}(x)$ obeying Eq.(4.9). The operator $\tilde{\mathcal{O}}(x)$ is called *primary*. Besides, a similar analysis of the fermionic operators leads to analogous primary operators.

To sum up: starting with the primary operator $\tilde{\mathcal{O}}(x)$, we can build new operators with the same dimension or higher, by repeatedly commuting it with D (Di Francesco et al. 1997). The higher-dimensional operators are called *descendants* of $\tilde{\mathcal{O}}(x)$.

4.4 Gauge invariant operators and a Fock space

We now apply the previous Subsection's discussion to the operators one actually encounters in SYM. The punchline at the end will be that the anomalous dimension of certain operators, which turn out to dominate in the large N limit (called *single trace* operators), will, for large N , be encoded in the Hamiltonian of a spin chain, in which each site carries a representation of $SO(6)$, the R -symmetry subgroup of $PSU(2, 2|4)$.

We recall that the physical observables of a gauge theory are gauge invariant operators. In SYM, the local gauge invariant operators are made up of products of traces of the fields that transform covariantly under the gauge group $SU(N)$. These fields include the scalars ϕ^I , the fermions ψ_α^a , $\bar{\psi}_{\dot{\alpha}a}$, the field strengths $\mathcal{F}_{\mu\nu}$ and their covariant derivatives. It is thus clear that the single trace local operator

$$\mathcal{O}(x) = \text{Tr}[\chi_1(x)\chi_2(x)\dots\chi_L(x)], \quad (4.10)$$

where the trace is over the internal degree of freedom indices, and $\chi_i(x)$ is one of the above covariant fields (with or without covariant derivatives), is itself gauge invariant. We can also build other local gauge invariant operators by taking products of traces.

In Section 5, we will take the 't Hooft limit, where the number of colours N is large. This limit has the remarkable property that the scaling dimension of the product of single trace operators is equal to the sum of their scaling dimensions, so that all information about the spectrum of local operators is determined by the single trace operators. Thus, for computing dimensions in this limit, it will suffice to concentrate on single trace operators.

Among the many remarkable properties of conformal field theories, one is a subtle correspondence between operators and states. (This was worked out in the 1980s; cf.

Belavin et al. (1983).) An example of this occurs for our single trace operators in SYM: viz. we represent operators as states in a Fock space built using bosonic and fermionic creation operators. We can build a field χ_ℓ ($\ell = 1, \dots, L$) within a single trace operator by applying to a ground state $|0\rangle$ appropriate elements drawn from two sets of bosonic creation operators A^\dagger , B^\dagger , and a set of fermionic creation operators $C^{a\dagger}$. Here, we define the operator

$$\mathcal{C} := A_\alpha^\dagger A^\alpha - B_\alpha^\dagger B^\alpha + C^{a\dagger} C_a - 2; \quad (4.11)$$

and the states that correspond to the actual fields are those states $|\chi\rangle$ in the Fock space for which $\mathcal{C}|\chi\rangle = 0$. This set of states, the image of \mathcal{C} , is itself a Fock space, which we denote by \mathcal{V} . For example, some states satisfying the $\mathcal{C}|\chi\rangle = 0$ condition, and the fields they correspond to, are:

$$\begin{aligned} (A^\dagger)^{k+1} (B^\dagger)^k C^{a\dagger} |0\rangle & \text{ corresponds to } \mathcal{D}^k \psi^a \\ (A^\dagger)^k (B^\dagger)^k C^{a\dagger} C^{b\dagger} |0\rangle & \text{ corresponds to } \mathcal{D}^k \phi^{ab}. \end{aligned} \quad (4.12)$$

All the generators of the total symmetry group, i.e. the superconformal group $PSU(2, 2|4)$, commute with \mathcal{C} ; so the symmetry group preserves the $\mathcal{C} = 0$ eigenspace.

In Section 5.2, we will return to the projected Fock space \mathcal{V} in more detail. But we can already state the key idea about how all this relates to spin chains. (We will postpone our philosophical classification of planarity and integrability until after Section 5's details.) For a single trace operator with L arguments, $\mathcal{O}(x) = \text{Tr}[\chi_1(x)\chi_2(x)\dots\chi_L(x)]$, we consider a spin chain of length L , each of whose sites carries a representation of the R -symmetry group $SO(6)$. Thus each site corresponds to one of \mathcal{O} 's arguments. And this correspondence is very informative. Not only do we have: a state in the Fock space at site ℓ , built up from the vacuum by a sequence of creation operators, corresponds to a field $\chi_\ell(x)$. But also: the anomalous dimensions of single trace local operators will, for large N , be encoded in the Hamiltonian of the corresponding spin chain.

5 Integrability at the limit, and the relation to spin chains

In Section 5.1, we will first outline the computation of anomalous dimensions: recall Section 1's old dream of computing a quantum field theory's mass spectrum, and Section 4.3's introduction to anomalous dimensions. More precisely: we discuss the one-loop anomalous dimensions for a general set of single trace operators; (recall from Section 4.4 that in the large N limit, the single trace operators encode all the spectral information). We will see how in the large N limit, the contributions to the anomalous dimensions are dominated by diagrams that can be drawn on a plane, like those discussed in Section 3.2.

Then in Section 5.2, we describe the mapping of the system into the problem of computing the energies of a certain spin chain with nearest-neighbour interactions. That is: the one-loop anomalous dimensions will be given by the eigenvalues of the corresponding spin chain’s Hamiltonian.

We stress that as usual, to characterise these anomalous dimensions, one adopts a perturbative scheme, e.g. an expansion in the ‘t Hooft coupling.²⁴ As to the history, we note that although related ideas about integrability in quantum field theory were discussed by Polyakov (1977), the results about SYM in this Section were mainly prompted by the seminal work of Minahan and Zarembo (2003).

5.1 Computing anomalous dimensions

In this Subsection we will concentrate on the one-loop anomalous dimensions for single trace operators composed of scalar fields ϕ^I with no covariant derivatives. (In fact, all operators can be built from those whose arguments contain no double derivatives.) Recall that the anomalous dimension is given by the exponent in the two-point correlator of the operator with itself, eq. 4.6. All scalar fields have classical dimension 1; and so for single trace operators made up only of scalar fields with no covariant derivative, the classical dimension of the operator is L , the number of arguments i.e. scalar fields inside the trace.

If the coupling constant g is small, then the anomalous dimension γ is much smaller than the bare dimension Δ_0 : $\gamma \ll \Delta_0$. In this case we can approximate the correlator in Eq.(4.6) as

$$\langle \mathcal{O}(x) \overline{\mathcal{O}}(y) \rangle \approx \frac{1}{|x - y|^{2\Delta_0}} (1 - \gamma \ln \Lambda^2 |x - y|^2), \quad (5.1)$$

where Λ is the cutoff scale. The leading, i.e. classical, contribution to this correlator, $1/|x - y|^{2\Delta_0}$, is called the ‘tree-level contribution’.

Let us now describe what happens as we let $N \rightarrow \infty$. We shall see that:

(1): the ideas in Section 3.2, about diagrams that can be drawn on a plane coming to dominate the expansion, occur here also (Section 5.1.1); and

(2): the anomalous dimension is given by an operator Γ —which in Section 5.2 will be the Hamiltonian of a spin chain (Section 5.1.2).

5.1.1 Planar diagrams dominate

We will consider as an example single trace operators for which: (i) all the arguments χ are the same field (so that the number of arguments $L \geq 2$ becomes a power); and

²⁴Returning to Section 2.2.1’s discussion whether a quantum field theory exists: note that for certain two-dimensional conformal field theories, the anomalous dimension γ can be computed in terms of a fractal dimension of a random walk by the Schramm-Loewner evolution (cf. Cardy (2005)). And the connection between two-dimensional conformal field theories and random walks goes far beyond computing anomalous dimensions. It may also provide a way to understand rigorously a class of interacting quantum field theories.

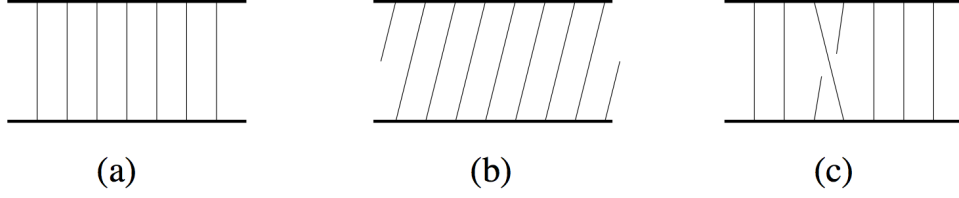


Figure 1: The horizontal lines represent the operators and the ordered vertical lines represent the contractions between the individual fields inside the trace. (a) and (b) are planar; while (c) is nonplanar, since lines must cross.

(ii) the common argument χ has trace zero. So the operator is $\text{Tr}[\chi^L]$, with $L \geq 2$ and $\text{Tr}\chi = 0$. It is often written Ψ_L for short. We rescale the operator as follows:

$$\Psi_L := \frac{(4\pi^2)^{L/2}}{\sqrt{L}N^{L/2}} \text{Tr}\chi^L = \frac{(4\pi^2)^{L/2}}{\sqrt{L}N^{L/2}} \chi^A{}_B \chi^B{}_C \cdots \chi^{\cdots}{}_A \quad A, B, C = 1, \dots, N, \quad (5.2)$$

where we have explicitly put in the colour indices. The prefactors are for normalization purposes. At tree level, the correlator of a χ field and its conjugate $\bar{\chi}$ is²⁵

$$\langle \chi^A{}_B(x) \bar{\chi}^C{}_D(y) \rangle_{\text{tree}} = \frac{\delta^A{}_D \delta_B{}^C}{4\pi^2 |x - y|^2}. \quad (5.3)$$

If we now contract Ψ_L with its conjugate $\bar{\Psi}_L$, then the leading contribution to the correlator comes from contracting the individual fields in order, as shown in Figure 1 (a) and (b). The contribution of all such ordered contractions is

$$\langle \Psi_L(x) \bar{\Psi}_L(y) \rangle_{\text{ordered}} = \frac{LN^L}{(\sqrt{L}N^{L/2})^2 |x - y|^{2L}} = \frac{1}{|x - y|^{2L}}. \quad (5.4)$$

The factor of N^L comes from L factors of $\delta^{A'}{}_A \delta^A{}_{A'} = N$, where each double set of delta functions is from contractions of neighbouring fields. The factor of L comes from the L ways of contracting the fields in the plane, of which (a) and (b) are two examples.

Figure 1 (c) is an example of a nonplanar diagram, a diagram where the lines connecting the fields cannot be drawn in the plane without cutting other lines. To avoid such cuttings one must lift at least one connecting line out of the plane. The diagram in (c) differs from (a) by two field contractions. Whereas in (a), we would have a factor of

$$\cdots \delta^{A'}{}_A \delta^A{}_{A'} \delta^{B'}{}_B \delta^B{}_{B'} \delta^{C'}{}_C \delta^C{}_{C'} \cdots = \cdots N^3 \cdots, \quad (5.5)$$

²⁵We have ignored the fact that, because the fields are in the adjoint representation, $\chi^A{}_A = 0$: this is justifiable when we take the large N limit.

in (c) we have the factor

$$\dots \delta^{A'}_A \delta^A_{B'} \delta^{C'}_B \delta^B_{A'} \delta^{B'}_C \delta^C_{C'} \dots = \dots N \dots, \quad (5.6)$$

where the dots represent contractions that are the same in both cases. Hence, the nonplanar diagram in (c) is suppressed by a factor of $1/N^2$ compared to that in (a). In the limit $N \rightarrow \infty$, we can thus ignore this contribution compared to the one from (a) or (b).

One can show that this example, Ψ_L , is typical. That is: all nonplanar diagrams will be suppressed by powers like $1/N^2$, where the power (here, 2) depends on the topology of the diagram.²⁶

Returning to our classificatory project: let us classify the property, planarity, for SYM, as we did for QCD in Section 3.2. There are three points to make: the first is an echo of our previous classifications, but the second and third will be unlike QCD.

(i): First: The situation is as it was for QCD (and as we announced in (b) of Section 2.2.2), in the following sense. We have just seen that as N grows, the planar diagrams come to dominate; and in the limit $N \rightarrow \infty$, only planar diagrams contribute. So again we have Norton's case (1), assuming that SYM rigorously exists in the 't Hooft limit.

(ii): Second: Unlike QCD, where we know little about the dynamics of the emergent string-like structures, here in SYM we know a good deal about them. For it turns out that they are the strings of one of the best-studied string theories, viz. type IIB string theory defined on the 10-dimensional spacetime $AdS_5 \times S^5$: which is the paradigm example of the AdS/CFT correspondence mentioned in Section 1. This returns us to the discussion of strings within QCD in remark (2) in Section 3.2.1. The dynamics of these required the introduction of extra dimensions (recall Polyakov's Liouville dimension). In the context of AdS/CFT, these extra dimensions correspond to the energy scale of the renormalisation group flow; (for a recent discussion, cf. Heemskerk and Polchinski (2011)).

(iii): Third: The emergent planarity in SYM has been shown to be associated with integrability—as we will see below. This has not been shown for QCD, although QCD may be integrable in the limit—and one of course hopes so. For a recent discussion, cf. Belitsky et al. (2004).

5.1.2 The operator Γ

Without going into details, we report that the one-loop anomalous dimension γ is encoded in an operator, Γ , whose eigenvalues are γ . (This arises from operator mixing.)

²⁶Actually, this analysis is valid only if $L \ll N$. If L were of the order of N then the suppression coming from the $1/N$ factors is swamped by the huge number of nonplanar diagrams compared to the number of planar diagrams. For there are $L!$ total tree-level diagrams of which only L are planar.

The definition of Γ is a sum of terms, which involve: (i) an exchange operator $P_{\ell,\ell+1}$ which, as its name suggests, exchanges the flavour indices of the ℓ and the $\ell + 1$ arguments inside the trace in Eq.(5.2); and (ii) a trace operator $K_{\ell,\ell+1}$ which contracts the flavor indices of fields at neighbouring arguments. To be precise: the action of these operators on a sequence of δ -functions, with indices I and J labelling the arguments inside (the trace of) our single trace operator, is

$$P_{\ell,\ell+1} \delta_{I_1}^{J_1} \dots \delta_{I_\ell}^{J_\ell} \delta_{I_{\ell+1}}^{J_{\ell+1}} \dots \delta_{I_L}^{J_L} = \delta_{I_1}^{J_1} \dots \delta_{I_\ell}^{J_{\ell+1}} \delta_{I_{\ell+1}}^{J_\ell} \dots \delta_{I_L}^{J_L} . \quad (5.7)$$

and

$$K_{\ell,\ell+1} \delta_{I_1}^{J_1} \dots \delta_{I_\ell}^{J_\ell} \delta_{I_{\ell+1}}^{J_{\ell+1}} \dots \delta_{I_L}^{J_L} = \delta_{I_1}^{J_1} \dots \delta_{I_{\ell+1}}^{J_{\ell+1}} \delta_{I_{\ell+1}}^{J_\ell} \dots \delta_{I_L}^{J_L} . \quad (5.8)$$

In particular, for operators made up of scalar fields: Γ is given by the formula

$$\Gamma = \frac{\lambda}{16\pi^2} \sum_{\ell=1}^L (1 - C - 2P_{\ell,\ell+1} + K_{\ell,\ell+1}) , \quad (5.9)$$

with λ is the 't Hooft coupling, and C a constant arising from a certain set of diagrams. And the possible one-loop anomalous dimensions can then be found by diagonalizing Γ .

5.2 Spin chains

Recall that at the end of Section 4.4, we built a field χ_ℓ ($\ell = 1, \dots, L$) within a single trace operator with L arguments, by using appropriate creation operators, giving a Fock space \mathcal{V} , satisfying a constraint $\mathcal{C}|\chi\rangle = 0$ and preserved by the symmetry group, i.e. the superconformal group $PSU(2, 2|4)$.

Indeed, the states in this Fock space, such as (4.12), form an irreducible representation of $PSU(2, 2|4)$, called the ‘singleton’ representation. However, for the single trace operators on which we have focussed since Sec 5.1.1 (i.e. a traceless common argument χ), this cannot be a representation of all gauge invariant operators since all of the fields corresponding to these states are traceless. Hence we will need $L \geq 2$ fields inside the trace, leading to tensor products of the singleton representation:

$$\mathcal{V}_1 \otimes \mathcal{V}_2 \otimes \dots \otimes \mathcal{V}_L, \text{ with } \mathcal{V}_1 \cong \mathcal{V}_j \cong \mathcal{V} . \quad (5.10)$$

And the various generators of $PSU(2, 2|4)$ on the tensor product have the general form

$$\mathcal{T} = \sum_{\ell=1}^L \oplus \mathcal{T}_\ell , \quad (5.11)$$

where \mathcal{T}_ℓ is the generator at site ℓ . We can also define \mathcal{C} in this way: however the projection is still carried out at each site, i.e. $\mathcal{C}_\ell|\chi_\ell\rangle = 0$. A gauge invariant operator

is then mapped to a state in the tensor product, but because of the cyclicity of the trace, it must be projected onto only those states that are invariant under the shift,

$$\mathcal{V}_1 \otimes \mathcal{V}_2 \otimes \cdots \otimes \mathcal{V}_L \rightarrow \mathcal{V}_L \otimes \mathcal{V}_1 \otimes \cdots \otimes \mathcal{V}_{L-1}. \quad (5.12)$$

Now we are ready for the punchline. In a very impressive collective effort,²⁷ it has been shown that the entire class of scalar single trace operators of length L can be mapped to a Hilbert space of a spin chain, i.e. a tensor product of *finite*-dimensional Hilbert spaces

$$\mathcal{H}_1 \otimes \mathcal{H}_2 \cdots \otimes \mathcal{H}_\ell \otimes \cdots \otimes \mathcal{H}_L. \quad (5.13)$$

Here each \mathcal{H}_ℓ is the Hilbert space for an $SO(6)$ vector representation. In other words: the Hilbert space is that of a one-dimensional spin chain with L sites, where at each site there is an $SO(6)$ vector “spin”. Because of the cyclicity of the trace, we should include the further restriction that the Hilbert space be invariant under the shift

$$\mathcal{H}_1 \otimes \mathcal{H}_2 \cdots \otimes \mathcal{H}_\ell \otimes \cdots \otimes \mathcal{H}_L \rightarrow \mathcal{H}_L \otimes \mathcal{H}_1 \cdots \otimes \mathcal{H}_{\ell-1} \otimes \cdots \otimes \mathcal{H}_{L-1}. \quad (5.14)$$

The operator Γ in (5.9) acts linearly on this space. Furthermore, it is Hermitian and commutes with the shift in (5.14). It also turns out that we can treat Γ as a Hamiltonian on the spin chain. The energy eigenvalues then correspond to the possible anomalous dimensions for the scalar operators. Since the Hamiltonian commutes with the shift, it is also consistent to project onto eigenstates that are invariant under the shift. Because $P_{\ell,\ell+1}$ and $K_{\ell,\ell+1}$ in (5.14) act on neighbouring fields, the spin chain Hamiltonian only has nearest-neighbour interactions between the spins.

Although we will not show it here, the Hamiltonian that corresponds to Γ for the spin-chain is integrable. Integrability is a subtle idea with various aspects: in particular, varying across classical and quantum theories. The core classical idea is that the number of conserved charges equals the number of physical degrees of freedom. This even applies to systems with infinitely many degrees of freedom, e.g. classical sine-Gordon theory: it has an infinite number of conserved charges. For SYM, one needs to take account of quantum aspects of integrability; (the idea of quantum integrability was discussed by Polyakov (1977)). But in short: an integrable infinite quantum system is described by an infinite number of commuting scalar charges (Frishman and Sonnenschein (2010, Chap 5)); and SYM’s integrability turns out to be related to the classical integrability of the corresponding string-like structures described by a sigma model. The situation is summarized in Table 1, which is based on Frishman and Sonnenschein (2010, p. 332).

Finally, we stress that our discussion has been confined to one loop calculations. Going beyond one loop, one finds that the n -loop contribution to the anomalous dimension can involve up to n neighboring fields in an effective Hamiltonian. Therefore, as N increases and the ‘t Hooft coupling λ becomes larger, these longer-range interactions become more important; so that at strong coupling the spin-chain is effectively

²⁷For a review we recommend Beisert et al. (2012) and Beisert (2005).

Table 1: Integrability at the limit

$\mathcal{N} = 4$ SYM at the limit	Integrable spin chain
Single trace operator	Cyclic spin chain
Field operator	Spin at a site
Anomalous dilatation operator $\delta D(g)$	Hamiltonian
Anomalous dimension γ	Energy eigenvalue

long-range. In this case, the Hamiltonian is not known above the first few loop orders.

Again, we summarize in terms of our classificatory project. At first, the classification of integrability looks similar to the classification of the previous property, planarity. That is, one expects to say: if SYM rigorously exists in the 't Hooft limit, integrability is novel, emergent and a Nortonian case (1). But beware. Agreed: it is novel, emergent and yet reduced. But it is reduced to properties different from its “cousin-property”, integrability at finite N . Thus as we warned in Section 2.2.2 (especially (c)): so far as is known at present, integrability is an illustration of Norton’s case (3). The reason is that it is hard to ascertain at finite N integrability in this Section’s sense. We will take up this topic in Section 6.

But first, let us briefly discuss integrability’s significance. One main significance is that it sheds light on the conjectured AdS/CFT correspondence. Indeed, although our exposition has not stressed the fact: most of the results reviewed above have used, or been inspired by, string-theoretic ideas and results; and often, ideas and results about AdS/CFT. And conversely, there is hope that these results will help prove the conjectured correspondence. It is a strong/weak duality: the strong coupling regime of a conformal field theory (CFT) corresponds to a weak, even classical, regime of a gravity theory. So one would expect such a correspondence to be very difficult to confirm, since one theory is computationally under control only when the other is not. But integrability of the conformal field theory means we can calculate a good deal about it at strong coupling—and so gather the necessary evidence for the conjecture, at least in the 't Hooft limit. Besides, integrability of SYM is especially enlightening, since this CFT (together with, on the gravity ‘side’, type IIB string theory on the 10-dimensional spacetime $AdS_5 \times S^5$) forms the best-understood case of the correspondence.

Though many questions remain open, there is reasonable hope that these integrability results will teach us how to go back to the physically relevant case of QCD, and finally arrive at the long-sought dual description of it by a string theory. It may even take us closer to realizing the quantum field theorist’s ultimate goal, unfulfilled for more than eighty years: completely understanding an interacting relativistic quantum field theory in the four space-time dimensions that we are familiar with.

6 Aspects of non-integrability before the limit

As announced already in Section 2.2.2: the first two of our theories’ three properties, beta-function behaviour and planarity, do not require a further discussion of their behaviour before the limit. But on the other hand, integrability does.

The previous sections have focussed on one particular case of integrability: the ‘success story’ of solving the exact planar (i.e. $N = \infty$) sector of SYM. While this case has been at the centre of attention, many investigations have tried to extend integrability to: (i) the finite N case; and (ii) other more realistic theories, like QCD. We will briefly report some attempts directed at (i). For brevity, we consider only SYM: we set aside (ii), other theories—for which, cf. Belitsky et al. (2004).

One general reason why we would expect integrability to fail at finite N is that integrability would seem to imply conservation of particle number in any scattering process: which would certainly not be expected in a four-dimensional gauge theory (Frishman and Sonnenschein 2010, Chapters 5, 18). But even if, accordingly, one focusses on some sector or regime of the finite- N theory, not much is known about integrability. The finite- N version of the dilatation generator can be written down easily enough (at least in some sub-sectors and to a certain loop order). But attempts to diagonalize it have so far not revealed any traces of integrability.

We turn to sketching some details about the difficulties involved. As we have seen in Section 5.1, planar SYM is described by only one parameter, the ‘t Hooft coupling λ , and planar anomalous dimensions have a perturbative expansion in terms of this single parameter. It is this fact that made it possible to search initially for integrability in the planar spectrum, by working order by order in λ . Thus the concept of ‘perturbative integrability’ was introduced: meaning that at l loops, i.e. disregarding terms of order λ^{l+1} , the planar spectrum could be described as an integrable system.

Accordingly, in studying the question of integrability at finite N , i.e. before the ‘t Hooft limit, it is natural to follow a similar perturbative approach, expanding in $1/N$. But so far, this approach has not borne fruit. Thus going beyond the ‘t Hooft limit seems to require some non-perturbative way of treating topologies with more handles etc., e.g. tori.

As an example of a simple way of getting evidence whether integrability persists at finite N , one can test for so-called degenerate parity pairs (Beisert et al. 2003; Section 7). This works as follows. Parity pairs are operators with the same anomalous dimension but opposite parity, where the parity operation on a single trace operator is defined by

$$P \cdot \text{Tr}(X_{i_1} X_{i_2} \dots X_{i_n}) = \text{Tr}(X_{i_n} \dots X_{i_2} X_{i_1}). \quad (6.1)$$

(For a multi-trace operator, P must act on each of its single trace components.) In the ‘t Hooft limit, at one-loop, there are a lot of such parity pairs. The presence of these degeneracies has its origin in the integrability of the model. $\mathcal{N} = 4$ SYM theory is parity invariant and its dilatation generator commutes with the parity operation,

i.e.

$$[D, P] = 0. \quad (6.2)$$

Note that this only tells us that eigenstates of the dilatation generator can be organized into eigenstates of the parity operator and nothing about degeneracies in the spectrum. The degeneracies imply the existence of an extra conserved charge, Q , which commutes with the dilatation generator but anti-commutes with parity, i.e.

$$[D, Q] = 0, \quad \{P, Q\} = 0. \quad (6.3)$$

Acting on a state with Q , one obtains another state with the opposite parity but with the same energy. Taking into account non-planar corrections, we expect that the degeneracies are lifted. Since parity is still conserved, this indicates (though of course, it does not prove) the disappearance of the extra conserved charges, and a breakdown of integrability.

To sum up: it seems that there is little hope for integrability of the spectrum of these field theories before the 't Hooft limit, at least in this simple perturbative sense, i.e. by expansion in the coupling constant λ .

In closing, we mention that accordingly, one turns to consider quantities other than anomalous dimensions. These include local operators such as structure constants, and non-local ones such as Wilson loops, 't Hooft loops, surface operators and domain walls. But the evaluation of structure constants is complicated because of operator mixing. Turning to non-local quantities: in fact, before the spin chain integrability that we have reviewed was discovered, it was already known that the expectation values of Maldacena-Wilson loops can in certain cases be expressed in terms of expectation values of a zero-dimensional integrable matrix model. Besides, this connection still constitutes one of the most successful tests, so far, of the AdS/CFT correspondence beyond the planar limit. But the relation of Maldacena-Wilson loops to spin chain integrability is so far not understood.

7 Conclusion

Our project in this paper has been to relate the 't Hooft limit to philosophical discussion of inter-theoretic relations. More specifically, we have classified the behaviour in this limit of three properties of two theories, QCD and SYM. To classify them, we used a schema of Butterfield's and a trichotomy of Norton's. Our verdict was that the properties mostly illustrate Butterfield's notion of emergence with reduction; and in Norton's trichotomy, mostly his case (1), called 'idealization'. But we will not here give a longer summary: we already gave one, by way of orienting the reader, in Sections 1 and 2, especially Section 2.2.2.

It is clear that the 't Hooft limit is a rich subject, and we have only scratched the surface—or, if you prefer, opened Pandora's box. So we will end by stating two topics we have not addressed. The first is physical, the second is philosophical.

First: one might use the 't Hooft limit to understand the mass-gap in quantum gauge theories, especially QCD. This is widely recognized to be the most important open theoretical problem about these theories (Polyakov (1987, Chapter 1.8), Jaffe and Witten (2006), Witten (2003, 2008)). Intuitively speaking, the problem arises from the fact that a classical non-abelian gauge theory has solutions describing waves obeying a non-linear wave-equation. But, unlike classical electromagnetism with its linear wave-equation, we do not see any such waves. The property of the quantum theory that is responsible for this is the mass-gap: every excitation of the vacuum has an energy of at least M , with $M > 0$. So the problem is to understand why the quantum theory has this property, while the classical one does not. (This is unlike the origin of mass in the electroweak theory, which is usually to add a scalar field, the Higgs boson.) For QCD itself, it has often been suggested that the 't Hooft limit may provide our best route for solving the problem (Polyakov (1987, Chapter 8.4); Witten (2003, p. 25); cf. 1998).²⁸

Second: there is a cluster of logical and philosophical issues, under headings such as 'the equivalence of theories', 'analogy' and 'duality'. We have not here articulated these issues, but it is clear that they have been illustrated in several ways. The obvious main case is the correspondence between SYM at $N = \infty$ and spin chains. We have also: (i) occasionally compared the 't Hooft limit with the thermodynamic limit, and (ii) occasionally mentioned AdS/CFT, which is also known as 'gravity-gauge duality' (cf. the end of Section 5.2). But a detailed treatment of these issues for the 't Hooft limit is work for another day.

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²⁸Incidentally: here lies an analogy with the thermodynamic limit's role in understanding phase transitions. There, it is surely a sufficient answer to the sceptic's objection that his boiling kettle has finitely many atoms, to say that the limit helps us understand the boiling. Similarly here: it is surely a sufficient answer to the sceptic's objection that nature no doubt uses only finitely many colours, to say that the 't Hooft limit may help us understand the mass-gap.

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